ELEMENTARY COURSE

IN

LAGRANGE'S EQUATIONS

AND

THEIR APPLICATIONS TO SOLUTIONS OF PROBLEMS OF DYNAMICS

WITH NUMEROUS EXAMPLES

BY

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PREFACE.

In compiling this work I have freely drawn on all best known books of such authorities as Appell, Routh, Loney, Bouasse and many others. My object is to present the matter in an elementary form and to make it immediately intelligible to a reader possessing but a limited knowledge of mathematics. For this reason it was thought necessary to explain things, already almost evident; constantly to refer back to elementary books on mechanics, etc.; and to devote the whole first chapter to elements, which are no doubt already familiar to the average reader, but which he may find presented in a somewhat different form from that in which he has them fixed in his mind.

Lagrange's method is like a slide rule: it has its limitations, yet, in many problems it enables us to write down the differential equations of motion almost instantly.

The wonderful beauty and power of this method will undoubtedly appeal to the reader, engineer or student, and make him *like* the whole subject of dynamics, although his teachers may have completely failed even to *interest* him in it, as often is the case, beyond the painful necessity of memorizing a few distorted notions.

However the primary object of the book is to be used in everyday practice; the writer, being but an average engineer, uses this method to great advantage in working out various problems of construction, etc. Why not suppose that others might likewise derive some benefit from this brief exposition of its principles? Those who want to know more are referred to *Appell,* Mécanique Rationnelle, Vol. II, and *Routh,* Dynamics of Rigid Bodies, Vols. I and II.

My thanks are due to Dr. Eric Doolittle, Director of the Flower Astronomical Observatory, for reading the MSS. and making many valuable suggestions.

N. W. A.

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CHAPTER I.

BRIEF SYNOPSIS OF CERTAIN PRINCIPLES OF DYNAMICS.

1. Constraints. By constraints special conditions are meant, limiting the motion of a particle in a certain manner, prescribed beforehand. For instance a particle may be free to move *only* along a certain curve (a small ring sliding on a curved wire, a car on the track, etc.); or, again, the particle may be compelled to remain, at all times, in contact with a certain surface (for instance if connected by means of a rod to a fixed point, about which it can, therefore, move on a sphere). Very often, the particle can move only on the exterior of a certain surface (imagine, for instance, a small particle sliding off a circular log, a well-known problem); or the distance between a particle and a certain fixed point may be prescribed to be equal to or less than a certain value (case of a stone on a string). All these are typical instances of *constrained motion.*

Analytically, the constraints are specified by geometric equations. For instance, the surface on which the particle is compelled to remain is usually given by some such equation as $f(x, y, z) = 0$; the curve, along which the particle can slide, would be given as the intersection of two surfaces such as $f(x, y, z) = 0$, and $F(x, y, z) = 0$.

It is of the utmost importance to note that the constraints may be either permanent or movable; that is, changing their position or even their shape. Consider, for instance, the motion of a particle constrained to move in a plane which itself is rotating, say, about a vertical axis with a certain angular velocity; or, the motion of a small particle of dust upon a soap bubble while it is being inflated. Conditions of this sort are characterized by the fact that the equations of

constraints contain the *time;* that is, the equation of a surface would then be $f(x, y, z, t) = 0$; and the constraining curve would be given by such equations as $f(x, y, z, t) = 0$; and $F(x, y, z, t) = 0$; so that the derivative with respect to time would then not equal 0. If it is 0, this means that the constraints are permanent, or independent of time.

In the absence of constraining conditions the motion of a particle is termed *free.*

2. Virtual work. The fundamental conception of virtual work and virtual velocity is known from elementary treatises (Bowser, Anal. Mech., p. 166). By virtual displacement we shall understand a very small displacement of a particle, *conceived* or *imagined* by us to take place in any direction whatsoever; it may or may not coincide with the displacement actually taking place under the action of the given forces and other conditions; the latter is called *actual* displacement. In case of constrained motion, certain displacements, called *compatible* or *consistent* with the constraints, can be conceived. For instance, in the case of a constraining curve the only compatible displacement would be either backward or forward, from some initial position, along the curve; in the case of a constraining surface, compatible displacements of a particle can be imagined to take place in a great variety of manners, but always subject to the initial condition, viz., adhesion to the surface. Other displacements cannot even be conceived without calling into play the idea of *distorting* the constraints; they are called *inconsistent* with the constraints and will not here be considered; while under free or unconstrained motion the virtual displacements may be any.

For the sake of clearness let us write down the few fundamental principles and definitions established so far: (*a*) By virtual work of a force is meant the product of the virtual displacement of its point of application into the projection of the force upon the direction of the displacement, in other words the virtual work $= P \cdot \delta p \cdot \cos(P, \delta p)$; (*b*) The virtual

work of a force for any displacement is equal to the sum of virtual works done by its components; in other words,

$$R \cdot \delta s \cdot \cos (R, \delta s) = \Sigma P \cdot \delta p \cdot \cos (P, \delta p);$$

(*c*) For concurring forces the sum of virtual works done by the forces equals the virtual work done by their resultant; from this is derived the very important form in which virtual work is given in rectangular coordinates (x, y, z). Supposing that there is a force P referred to rectangular axes x, y, z and that the projections of the force upon these axes are (say) X_0, Y_0 and Z_0. Let the virtual displacement of the force be δp, of which the projections upon the axes will be δx, δy, δz. Now in view of what has just been said, the virtual work of the force must equal the sum of virtual works of its components; that is,

$$P \cdot \delta p \cdot \cos (P, \delta p) = X_0 \delta x + Y_0 \delta y + Z_0 \delta z;$$

we will represent this simply by δW; (*d*) From (*c*) it also follows that when any number of concurring forces are in equilibrium the sum of their virtual works is $= 0$. (*e*) To the above the following principle should be added: In case of rotation the virtual work is the product of the moment of the force about the axis of rotation by the angular (virtual) displacement.

All of this refers to free motion. So far as *constrained* motion is concerned the following remarks may be made: If the motion of a particle is constrained, this of course means that at any time the coordinates of the particle must satisfy the constraining equation

$$f(x, y, z) = 0; \qquad \text{or,} \qquad f(x, y, z, t) = 0, \tag{1}$$

otherwise the particle would not remain on the constraining curve or surface; and if a small virtual displacement, of which the projections upon the axes are δx, δy, δz, be given to the particle, compatible with the constraints, then the new posi-

tion of the particle will still satisfy the equations (1). That is, if the original coordinates of a particle were x, y, z, satisfying the constraining condition (1), then the new coordinates, $x + \delta x$, $y + \delta y$, $z + \delta z$, must also satisfy the condition (1) for the same instant t (that is t remaining constant). The $f(x, y, z, t)$ must therefore be the same as $f(x + \delta x, y + \delta y, z + \delta z, t)$, and both of them must $= 0$; thus

$$f(x + \delta x, y + \delta y, z + \delta z, t) - f(x, y, z, t) = 0. \tag{2}$$

We can place the last equation $f(x + \delta x, y + \delta y, z + \delta z)$ in a somewhat more convenient form. The rules of elementary calculus will enable us to develop this function into a Taylor's series, that is in terms of the virtual increments, δx, δy and δz. Confining ourselves to the first power of the small increments we have

$$f(x + \delta x, y + \delta y, z + \delta z, t) = f(x, y, z, t) + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z. \tag{3}$$

Comparing this with (2) we see that if the displacement is consistent with the constraints, then

$$\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z = 0$$

must necessarily be zero; this is often written thus

$$\delta f = 0. \tag{4}$$

In this deduction the time t was taken as constant since (4) must hold true for any moment, that is must be absolutely independent of the time variation; and of course the constraints might have been independent of the time in the first place, that is instead of $f(x, y, z, t) = 0$, we might have had $f(x, y, z) = 0$, in which case the independence of the time would have been implied from the first. This may

seem a very obvious remark, but it should be well mastered by the reader.

Now *actual* displacement (not *virtual*) must take place consistently with changing constraints (that is, changing if they contain time). In general therefore the virtual displacements δx, δy, δz, will not be the same as the actual displacements dx, dy, dz; they will be the same only if the constraints are independent of the time, at least explicitly so; that is, if

$$\frac{\partial f}{\partial t} = 0; \qquad \frac{\partial F}{\partial t} = 0, \qquad \text{etc.}$$

3. D'Alembert's principle. This perfectly general and very powerful principle is seldom satisfactorily explained; yet it is quite easy to grasp and to apply it in practice. As a simple illustration of it consider a material particle moving along a constraining curve (fig. 1) under the action of some external

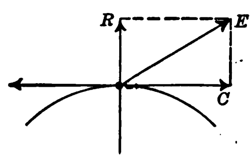


FIG. 1.

force E; this is usually called an *impressed* or *applied* force. It is quite evident that only a certain part of the force, E, will really act upon the particle in a manner consistent with the constraints. Resolving the impressed force E into two components, C, tangent and R, normal to the curve, we can say that, owing to the reaction of the constraints, the force R is lost or wasted, so far as the motion is concerned; and only the remaining force C (which is called the *effective, active* or *conserved* force), is able to produce the motion of the particle. We can readily see, therefore, that the effect of the impressed force E and that of the effective force C is precisely the same; namely, they both are capable of producing the same accelera-

tion and this is equal to the conserved force C divided by the mass of the particle. The only difference is that the force E has a certain effect on the constraints in form of the force R, taken up by the reaction of the constraint, and does not in the least affect the motion; while the force C has no effect whatever on the constraints.

Let us now imagine that the particle is acted upon by the same force E as before, and that two equal and opposite forces (fig. 2), C and C', have been applied to it as well. These

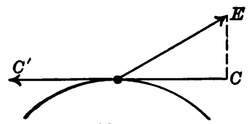


FIG. 2.

two latter forces are in equilibrium, and therefore do not affect the motion in any manner: under the action of the three forces the motion will be precisely the same as it would have been under the action of the force E alone. Since C and E are interchangeable so far as the motion is concerned, if C balances C', it is quite evident that E will likewise balance that force C'. Now C' is merely the reversed force C and equals $-C$ (it is often called the *force of inertia* or the *kinetic reaction*). This is stated in D'Alembert's principle: *In view of the constraints, the impressed forces are in equilibrium with the reversed effective forces.*

In extending this principle to a system (instead of a mere particle) the same reasoning can be applied to each separate particle of the system (or of a rigid body); and what we meant by constraints and represented as a curve to which the motion of the particle is confined, need not necessarily be a material path or track; it may also be construed to mean the action of one or several surrounding particles, or, for instance, the action of the rest of the system upon the specific particle under consideration. But the action of the surrounding particles upon the given particle is necessarily

equal to the action exerted by the latter upon the former; so that in a system all internal constraints, molecular actions, etc., automatically balance each other. The only forces which are unbalanced, and therefore produce the motion, are the impressed forces, or rather their effective components. In order to balance these we mentally apply the forces of inertia; that is, forces equal but acting opposite to the effective forces; this will produce equilibrium and thus reduce the problem from one of dynamics to one of statics. In other words, given all the impressed forces, we have but to calculate what forces, consistent with the system, would balance them; and then, reversing the latter, we will have the forces actually producing the motion. That is, given E we calculate C' to balance the former; then the reversed C' gives us the force that actually produces the motion and to find which is our ultimate object. Hence another way of expressing D'Alembert's principle: *If, at any time, the moving system be stopped and all impressed forces, as well as all forces of inertia, applied thereto, the system will remain at rest.*

Expressing this principle analytically, we can obtain the necessary equations of motion—our immediate aim. This is done in the following manner. If the impressed (or applied) forces are to be in equilibrium with forces of inertia (or reversed effective forces, that is—mj for each particle, j being the acceleration of motion), we can simply express this by stating that the virtual work done by such forces *combined* is 0 for any displacement consistent with constraints. The expression of virtual work in rectangular coordinates, as we have seen under *Virtual Work*, is $X_0\delta x + Y_0\delta y + Z_0\delta z$; but in our case any force X_0, for each particle, will consist of the projection of the corresponding impressed force, that is X, *plus* the projection on the same axis of the corresponding force of inertia, that is reversed effective force or mass times acceleration upon *that* axis; in other words

$$X_0 = X + \left(- m\frac{d^2x}{dt^2} \right);$$

and of course the same applies to y and z. We must write $= 0$, not the work of any one, but of all the particles of which the system is made up; this is indicated by the sign Σ, so that the final form of D'Alembert's principle, expressing analytically that there is at all times equilibrium between the impressed forces and the forces of inertia, is

$$\Sigma\left[\left(X - m\frac{d^2x}{dt^2}\right)\delta x + \left(Y - m\frac{d^2y}{dt^2}\right)\delta y + \left(Z - m\frac{d^2z}{dt^2}\right)\delta z\right] = 0. \tag{1}$$

This is the *fundamental equation of dynamics.* It can be applied to any system of particles or bodies; to free as well as to constrained motion (if free, δx, δy, δz are wholly arbitrary; if constrained, they must satisfy the equations of constraints). This equation was given by Lagrange. The differential equations of motion are directly derived from it. Let us suppose, for instance, that the system consists of n particles and that there are k constraining conditions

$$\begin{aligned} &f_1(x, y, z, x_1, y_1, z_1, \cdots t) = 0; \\ &f_2(x, y, z, x_1, y_1, z_1, \cdots t) = 0; \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \text{etc.} \end{aligned} \tag{2}$$

connecting the particles in a certain manner. These equations of constraints will, in general, contain $3n$ coordinates, $x, y, z, x_1, y_1, z_1, \cdots$ of the particles, as well as the time, t. The virtual displacements δx, δy, δz of the fundamental equation, in order to be consistent with the constraints, must satisfy such equations as $\delta f = 0$; in other words δx, δy, δz must satisfy the following k equations

$$\begin{aligned} &\frac{\partial f_1}{\partial x}\delta x + \frac{\partial f_1}{\partial y}\delta y + \frac{\partial f_1}{\partial z}\delta z + \frac{\partial f_1}{\partial x_1}\delta x_1 + \cdots = 0, \\ &\frac{\partial f_2}{\partial x}\delta x + \frac{\partial f_2}{\partial y}\delta y + \frac{\partial f_2}{\partial z}\delta z + \frac{\partial f_2}{\partial x_1}\delta x_1 + \cdots = 0, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{3}$$

(compare with (4) under *Virtual Work*). It will be observed that t has been assumed constant, since our discussion applies to any instant we wish to choose. From these equations (3) we can find any k such variables as δx, δy, δz, $\cdots$, etc., in terms of the other $3n - k$; introducing them into the fundamental equation we shall have only $3n - k$ variations to deal with, but they are now perfectly arbitrary since k characteristics due to constraints have been eliminated and therefore the constraints no longer enter into play. Now the remaining $3n - k$ variations being absolutely arbitrary, subject to no condition whatever, we can put each of their coefficients equal to 0, which is really the only way to satisfy the fundamental equation (1) for any arbitrary value of these variations. Therefore, equating to 0, all of the coefficients of the remaining $3n - k$ variations will give $3n - k$ differential equations of the second order, which, added to the k original equations of constraints will enable us to find the desired $3n$ coordinates as functions of time t, and this is precisely our object. In integrating the $3n - k$ differential equations of second order we shall introduce $6n - 2k$ constants which will be determined from the initial conditions of each problem.

In eliminating the k variations, δx, δy, δz, etc., from (1) by means of (3) it is much more convenient, instead of direct elimination, to use the following artificial method, which was also given by Lagrange: Each of equations (3) is multiplied throughout by a certain indeterminate factor such as λ_1, λ_2, $\cdots$, etc.; then they are all added to (1); factoring the result of such addition by δx, δy, δz, etc., we have

$$\Sigma\left[\left(X - m\frac{d^2x}{dt^2} + \lambda_1\frac{\partial f_1}{\partial x} + \lambda_2\frac{\partial f_2}{\partial x} + \cdots\right)\delta x \right.$$
$$+\left(Y - m\frac{d^2y}{dt^2} + \lambda_1\frac{\partial f_1}{\partial y} + \lambda_2\frac{\partial f_2}{\partial y} + \cdots\right)\delta y$$
$$\left. +\left(Z - m\frac{d^2z}{dt^2} + \lambda_1\frac{\partial f_1}{\partial z} + \lambda_2\frac{\partial f_2}{\partial z} + \cdots\right)\delta z + \cdots\right] = 0,$$

which equation involve $3n$ such variations as δx, δy, δz, δx_1, δy_1, etc. Here the coefficients of the first k variations δx, δy, δz, δx_1, δy_1, etc., can be made equal to 0 because the k arbitrary factors λ_1, λ_2, λ_3, may be *chosen* so as to satisfy this condition; and the remaining $3n - k$ variations being perfectly arbitrary, *their* coefficients must be all equal to 0, or else there could not be equilibrium, as given by the equation (1) under *all* conditions. In other words, all such expressions as

$$X - m\frac{d^2x}{dt^2} + \lambda_1\frac{\partial f_1}{\partial x} + \lambda_2\frac{\partial f_2}{\partial x} + \cdots,$$

$$Y - m\frac{d^2y}{dt^2} + \lambda_1\frac{\partial f_1}{\partial y} + \lambda_2\frac{\partial f_2}{\partial y} + \cdots,$$

etc. . . . . ,

(of which we have $3n$), must equal 0; these $3n$ equations together with the k equations of constraints will enable us to determine $3n$ coordinates and k multipliers λ_1, λ_2, $\cdots$, etc.

Taking one of such equations as for instance,

$$X - m\frac{d^2x}{dt^2} + \lambda_1\frac{\partial f_1}{\partial x} + \lambda_2\frac{\partial f_2}{\partial x} + \cdots = 0,$$

we can put it in the following form

$$m\frac{d^2x}{dt^2} = X + \lambda_1\frac{\partial f_1}{\partial x} + \lambda_2\frac{\partial f_2}{\partial x} + \cdots, \tag{4}$$

of which the extreme importance should not be overlooked. Remembering that *force equals mass times acceleration*, we can easily see that the right-hand side represents the sum of the forces (projected upon the corresponding axes, x, in this case); one of these forces is X, that is the projection of the given or impressed force; other terms represent forces due to the projected *reaction* of the constraints; so that this method is not only a powerful means for determining the motion proper, but gives the lost forces (or the corresponding reactions) as well.

We shall illustrate the application of both the fundamental equation and of the method of Lagrange's multipliers in the following easy example:

A particle of unit mass is constrained to move in a vertical plane which itself is made to revolve about a vertical axis with a uniform angular velocity ω. Find the motion.

Let the axis of rotation (fig. 3) be the axis of Z, directed downward, and the other axes—as shown. In the beginning

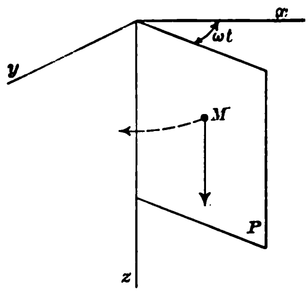


FIG. 3.

of motion, when the time t is 0, let the plane coincide with the plane x, z; then the plane will always be given by its angle ωt from its initial position, and its equation in rectangular coordinates will be $y = x \tan \omega t$. The equation of the constraint in our case will hence be

$$f(x, y, z, t) = y - x \tan \omega t = 0.$$

(This equation does not contain z, because the plane is parallel to the Z axis, but contains t, as a movable constraint should.) The only impressed or applied force is that of gravity, which for unit mass is $= g$; therefore we have in the fundamental equation (1) $X = 0$; $Y = 0$; $Z = g$.

In order to write down the equations similar to (4), into which the fundamental equation (1) resolves itself, we have to assume a certain Lagrange's multiplier λ (only one, since there is only one equation of constraint), so that the pro-

jected reactions of the constraints will be

$$\lambda\frac{\partial f}{\partial x}, \qquad \lambda\frac{\partial f}{\partial y}, \qquad \lambda\frac{\partial f}{\partial z}.$$

Differentiating $f(x, y, z, t)$, that is $y - x \tan \omega t$, with respect to the variables x, y, z, we have

$$\lambda\frac{\partial f}{\partial x} = -\lambda \tan \omega t; \qquad \lambda\frac{\partial f}{\partial y} = \lambda; \qquad \lambda\frac{\partial f}{\partial z} = 0;$$

so that the equations (4) will become (the mass being = 1)

$$\begin{aligned} \frac{d^2x}{dt^2} &= X + \lambda\frac{\partial f}{\partial x} = -\lambda \tan \omega t, \\ \frac{d^2y}{dt^2} &= Y + \lambda\frac{\partial f}{\partial y} = \lambda, \\ \frac{d^2z}{dt^2} &= Z + \lambda\frac{\partial f}{\partial z} = g. \end{aligned} \tag{a}$$

Integrating the last equation twice, we have the velocity and the height traveled in the time t:

$$\frac{dz}{dt} = gt + C_1; \qquad z = \frac{gt^2}{2} + C_1t + C_2,$$

where C_1 and C_2 are arbitrary constants. The first two equations can now be freed from λ, which can be done by means of the equation of constraints, as was suggested in the theory. Differentiating the first equation twice and substituting into the result the values (d^2x/dt^2) and (d^2y/dt^2) from (a) we have after easy reduction

$$\lambda = 2\omega\frac{dx}{dt} + 2\omega^2 x \tan \omega t$$

which, in the first equation (a) gives

$$\frac{d^2x}{dt^2} + 2\omega \tan \omega t \cdot \frac{dx}{dt} + 2\omega^2 \tan^2 \omega t \cdot x = 0. \tag{b}$$

It is of advantage to introduce here a new variable $x = r \cos \omega t$, whence

$$\frac{dx}{dt} = \frac{dr}{dt} \cos \omega t - r\omega \sin \omega t;$$

and

$$\frac{d^2x}{dt^2} = \frac{d^2r}{dt^2} \cos \omega t - 2\frac{dr}{dt} \omega \sin \omega t - r\omega^2 \cos \omega t.$$

Substituting the values of dx/dt and d^2x/dt^2 into (*b*) we have after reductions

$$\frac{d^2r}{dt^2} = r\omega^2,$$

of which differential equation the usual solution is

$$r = C_3 e^{\omega t} + C_4 e^{-\omega t};$$

but

$$r = \frac{x}{\cos \omega t}$$

so that

$$x = \cos \omega t (C_3 e^{\omega t} + C_4 e^{-\omega t}).$$

Having thus found both z and x in terms of the time we have completely determined the motion.

Another easy example illustrating D'Alembert's principle is as follows: *Two particles, P and P' (of mass m and m') are fastened to a light rod and made to swing like a pendulum. Find the motion.*

Let (fig. 4) $AP = a$ and $AP' = a'$. According to D'Alembert's principle we must imagine the system stopped; then there will be equilibrium under the action of the applied forces and the forces of inertia. In our case the applied forces are the weights mg and $m'g$ of the particles; their moments about A are $-\ mag \sin \theta$ and $-\ m'a'g \sin \theta$ (*minus*, because they decrease the angle; *moments*, instead of forces because one point of the system is fixed, which is always a hint). The forces of inertia are $m(d^2s/dt^2)$ and $m'(d^2s'/dt^2)$; and their

moments are $ma(d^2s/dt^2)$ and $m'a'(d^2s'/dt^2)$ (s and s' being the arcs described). Therefore

$$ma\frac{d^2s}{dt^2} + m'a'\frac{d^2s'}{dt^2} + (ma + m'a')g \sin\theta = 0.$$

But $ds = ad\theta$ and $ds' = a'd\theta$, so that

$$(ma^2 + m'a'^2)\frac{d^2\theta}{dt^2} + (ma + m'a')g \cdot \sin\theta = 0.$$

Comparing this with the problem of a compound pendulum (Bowser, Anal. Mech., p. 458) we see at once that the system

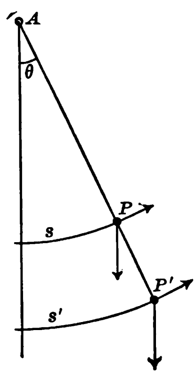


FIG. 4.

will oscillate precisely as a simple pendulum of the length

$$\frac{ma^2 + m'a'^2}{ma + m'a'}.$$

D'Alembert's principle and Lagrange's fundamental equation (1) are perfectly general and can be applied to any problem involving either equilibrium or motion. Several useful principles can be derived from these, which themselves help us to solve a great many problems, without applying the general equations. We shall mention only two such principles; that of *kinetic energy*, and of *areas*.

4. Integral of kinetic energy. The principle of *vis viva* or kinetic energy is already familiar from the elementary course (Bowser, Anal. Mech., p. 489), but it may easily be deduced from the fundamental equation of dynamics as follows: In estimating the value of the virtual work of a system we always referred it to some particular moment of time t, that is, we imagined the constraints momentarily fixed for that instant. The displacements in that case were denoted by δx, δy, δz, etc. But, if the constraints contain the time, then the real, actual, displacements dx, dy, dz, cannot, generally, be equal to the virtual displacements, δx, δy, δz, which were subject only to one condition, to be consistent with the constraints, temporarily fixed for that moment. For instance, if the constraints are represented by a surface A (fig. 5), we can

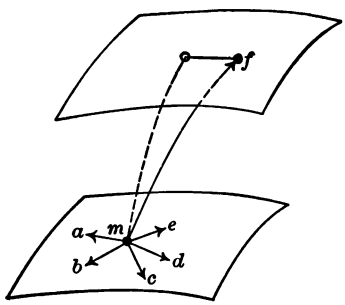


FIG. 5.

imagine a number of displacements $a, b, c, d, e, \cdots$, on that surface for a given instant, during which the surface is stationary; while in reality, owing to the motion of the surface itself, the actual displacement may be such as mf, that is entirely different from any virtual displacement we imagined for the case of a fixed surface. Omitting the case of movable constraints, however, that is supposing that the equations of constraints do not contain the time t, we can always select the virtual variations δx, δy, δz, etc., so that they will be equal to the actual displacements, dx, dy, dz, $\cdots$. In this

case (and in this case only) we have the fundamental equation of dynamics in the following form

$$\Sigma\left[\left(X - m\frac{d^2x}{dt^2}\right)dx + \left(Y - m\frac{d^2y}{dt^2}\right)dy + \left(Z - m\frac{d^2z}{dt^2}\right)dz\right] = 0,$$

which can be rearranged thus

$$\Sigma m\left(\frac{d^2x}{dt^2}dx + \frac{d^2y}{dt^2}dy + \frac{d^2z}{dt^2}dz\right) = \Sigma(Xdx + Ydy + Zdz). \quad (1)$$

The second member of this equation represents the *work*, not virtual but actual, done by the system (compare with the expression of virtual work given above). It is often possible to calculate a special function U of the coordinates (that is *only* of x, y, z, and *not* of the time or of time derivatives, etc.), of which the partial derivatives, with respect to axes, will give forces acting *along* these axes. Such a function is called the *potential function* and exists in a great many cases. If such a function exists, the right-hand side of (1) can be further simplified.

Indeed, from the very definition of the potential function we have

$$\frac{\partial U}{\partial x} = X; \quad \frac{\partial U}{\partial y} = Y; \quad \frac{\partial U}{\partial z} = Z;$$

which in the right-hand side of (1) gives

$$\Sigma\left(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz\right)$$

and this a total differential, dU, of the potential function U (Bowser, Calculus, p. 122). This will reduce the equation (1) to

$$\Sigma m\left(\frac{d^2x}{dt^2}dx + \frac{d^2y}{dt^2}dy + \frac{d^2z}{dt^2}dz\right) = dU.$$

In order to further simplify the first member we will use the well-known expression of velocity (Bowser, Anal. Mech., p. 244),

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2;$$

differentiating this with respect to time t we have

$$\tfrac{1}{2}d(v^2) = \left(\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2}\right) dt$$

$$= \frac{d^2x}{dt^2}\,dx + \frac{d^2y}{dt^2}\,dy + \frac{d^2z}{dt^2}\,dz;$$

so that finally we have $\frac{1}{2}d\Sigma mv^2 = dU$; integrating this we have

$$\Sigma m\,\frac{v^2}{2} - \Sigma m\,\frac{v_0^2}{2} = U - U_0 \qquad (2)$$

where the initial velocity v_0 and the initial value U_0 are given by the initial conditions of the problem. This result is known as *integral of kinetic energy;* it shows that the kinetic energy acquired (or lost) is equal to the difference of the initial and the final values of the potential (or force) function, no matter what path was followed in changing the position of the system. In other words, if the potential function exists, the increase (positive or negative) of the kinetic energy, between any two positions of the system, will be independent of the manner in which the change was performed.

Substituting

$$\Sigma m\,\frac{v^2}{2} - \Sigma m\,\frac{v_0^2}{2}$$

in equation (1) we have the expression

$$\Sigma m\,\frac{v^2}{2} - \Sigma m\,\frac{v_0^2}{2} = \int_{t_0}^{t} (Xdx + Ydy + Zdz), \qquad (3)$$

that is, the increase of kinetic energy equals the work done

by all the forces acting upon the system in the interval of time between t_0 and t; this applies also to the more general case, when there is no potential function.

By *all* the forces we mean all external, internal, applied or reaction forces; this latter expression applies whether the constraints are moving or not, to any system consisting of solid or liquid matter; it should be committed to memory by the reader merely as the *kinetic energy equals the work done.*

In our future work we shall often be called upon to calculate the kinetic energy of motion which is given, not in x, y, z, but in some other, less familiar, system of coordinates. In order to obtain this, all we have to do is to find x, y, z in terms of the new coordinates, then differentiate such expressions with regard to t and finally to substitute the results in

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2.$$

For instance, let the motion be given as taking place on a

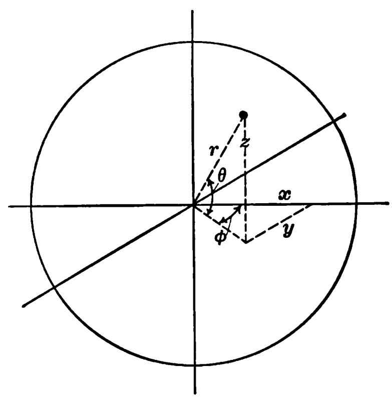


FIG. 6.

sphere of variable radius, r, and characterized, say, by latitude, θ, and longitude, φ. We can easily see that (fig. 6)

$$x = r\cos\theta\cos\varphi; \qquad y = r\cos\theta\sin\varphi; \qquad z = r\sin\theta,$$

and substituting the time-derivatives from these formulae into v^2 we have, after easy reduction,

$$m\frac{v^2}{2} = \frac{m}{2}\cdot\frac{dr^2 + r^2d\theta^2 + r^2\cos^2\theta d\varphi^2}{dt^2}.$$

In the case of plane polar coordinates, r, φ, this reduces (θ being 0) to

$$\frac{m}{2}\cdot\frac{dr^2 + r^2d\varphi^2}{dt^2};$$

or, in case of ordinary rotation (about an axis, when $r =$ const.) it further reduces to

$$\frac{m}{2}r^2\left(\frac{d\varphi}{dt}\right)^2 = I\frac{\omega^2}{2},$$

where I is the moment of inertia about the axis. This is a well-known expression of kinetic energy for rotation about a fixed axis.

To return to the integral of kinetic energy: it is called *integral* because it is a so-called *first integral* of motion; if put in the form

$$\Sigma m\frac{v^2}{2} = U + h$$

(where $\Sigma m(v_0^2/2)$ and U_0 are absorbed in one constant, h), we see at once that the second derivatives, d^2x/dt^2, etc., which appear in the fundamental equation, have vanished, and in the integral of kinetic energy we have only the variables $x, y, z, \cdots, (dx/dt), (dy/dt), (dz/dt), \cdots$, and *one* constant, h; such an expression itself is a partial solution of (1).

It is customary to denote kinetic energy by T, so that

$$dT = \Sigma(Xdx + Ydy + Zdz)$$

and, if there is a potential function, then

$$T = \Sigma m\frac{v^2}{2} = U + h.$$

This form of the integral of kinetic energy will be constantly employed. It has no meaning when the constraints are variable, because then there is no potential function U; but it exists in a great number of cases,—in all sorts of molecular, attractive (gravitational) forces, etc.

Let us take a few examples: For a particle under the action of gravity, the axis z being directed downward, the potential function will be $U = mgz$, because

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = 0,$$

as they should; while

$$\frac{\partial U}{\partial z} = mg,$$

that is the weight of the particle is the only moving force Z.

In all problems involving suspended or pivoting systems z is generally the vertical distance of the center of gravity from the fixed point (or axis).

If the axis z is directed upward, U would be $= -mgz$.

For a particle which is repelled inversely as the cube of the distance we can at once write down

$$U = -\frac{mk}{2x^2},$$

because the derivative of this expression with respect to the line of action, x, will give mk/x^3.

For a particle moving in a vertical plane, under the action of gravity, along a constraining curve, given in polar coordinates, r, θ, we have $U = mgz = mgr \sin\theta$ (if θ is reckoned from the horizontal axis x).

In problems involving turning moments (man turning a crank with a uniform moment M) the potential function will be

$$U = M \tan^{-1}\frac{z}{x}$$

(if y is the axis of rotation).

Indeed we have the following partial derivatives

$$\frac{\partial U}{\partial z} = -\frac{Mz}{x^2 + y^2}$$

and

$$\frac{\partial U}{\partial x} = \frac{Mx}{x^2 + y^2}.$$

Since $x^2 + y^2 = r^2$ (the radius of the crank) we see that these partial derivatives are equivalent to $-(M/r)\sin\varphi$ and $(M/r)\cos\varphi$, where φ is the angle of the radius with the axis x; in other words they represent *forces* along the axes z and x. This remark will be of practical interest in engineering work.

For a particle subject to the action of a force radiating from a center and proportional to the distance, we have

$$U = -\frac{ax^2}{2},$$

where a is a constant; because then

$$\frac{\partial U}{\partial x} = -ax,$$

as required.

Example. The motion of a system is given by the following equation

$$\frac{d^2x}{dt^2} = (\cos x - a)\sin x, \qquad \text{where } a > 0.$$

Find the potential function U.

Multiplying by $2(dx/dt)$ we have

$$2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = 2(\cos x - a)\sin x\frac{dx}{dt}.$$

Now, since the motion is evidently rectilinear, we have

$$T = \frac{mv^2}{2} = \frac{m}{2}\left(\frac{dx}{dt}\right)^2,$$

3

or, from the above,

$$T = \int (\cos x - a) \sin x dx + c = \tfrac{1}{2} \sin^2 x + a \cos x + c.$$

But, since $T = U + h$, we have the required solution

$$U = \tfrac{1}{2} \sin^2 x + a \cos x + b,$$

where b is a constant, absorbing c and h.

We shall say a few more words regarding the force function U and the practical methods of finding it.

Since by the very definition of the force function

$$\frac{\partial U}{\partial x} = X; \quad \frac{\partial U}{\partial y} = Y \quad \text{and} \quad \frac{\partial U}{\partial z} = Z;$$

therefore, in the general expression of elementary work we have

$$dW = Xdx + Ydy + Zdz = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz = dU;$$

that is *elementary work = complete differential of potential function.*

From this we should not conclude, however, that *any* function, of which the partial derivatives with respect to axes give projected forces on these axes, is the potential function, for this is not always the case; the following is the criterion: we have

$$\frac{\partial U}{\partial x} = X; \quad \frac{\partial U}{\partial y} = Y; \quad \frac{\partial U}{\partial z} = Z.$$

Differentiating the first equation as to y and the second as to x we have

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial X}{\partial y},$$

also

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial Y}{\partial x};$$

so that

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}.$$

Similarly

$$\frac{\partial X}{\partial z} = \frac{\partial Z}{\partial x};$$

also

$$\frac{\partial Y}{\partial z} = \frac{\partial Z}{\partial y}.$$

If these equations are identically satisfied, then (and then only) the potential function U exists and can be found by integrating the expression $Xdx + Ydy + Zdz$, giving, as we have already seen,

$$\frac{mv^2}{2} - \frac{mv_0^2}{2} = U - U_0,$$

which of course is nothing more than

$$\frac{mv^2}{2} - \frac{mv_0^2}{2} = \int (Xdx + Ydy + Zdz).$$

Example. If, for instance, the acting force is merely gravity, the axis z being directed downward, then $X = 0$; $Y = 0$; $Z = mg$. Here our criterion is identically satisfied and $dU = mgdz$, hence $U = mgz + c$; the constant, c, as a rule is taken $= 0$; although later on it will be shown that at times it may be convenient to assign it a certain value, in accordance with the problem (see *small oscillations*).

Example. A particle, x, y, z, is acted upon by a force, emanating from a center, the intensity being a function $f(r)$ of the distance from the center. Find the potential function.

Let us take the center as origin of coordinates. Then, the distance being $= r$, the cosines of r with the axes will be $-x/r$, $-y/r$, $-z/r$; and if the force is $= f(r)$, then its projections on the axes will be of course

$$X = -f(r)\frac{x}{r}; \qquad Y = -f(r)\frac{y}{r}; \qquad Z = -f(r)\frac{z}{r};$$

and therefore

$$Xdx + Ydy + Zdz = -\frac{f(r)}{r}(xdx + ydy + zdz).$$

However, $x^2 + y^2 + z^2 = r^2$, so that $xdx + ydy + zdz = rdr$; and

$$Xdx + Ydy + Zdz = -\frac{f(r)}{r} \cdot rdr = -f(r)dr = dU;$$

so that

$$U = -\int_0^r f(r)dr$$

is the potential function, if only

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}, \qquad \text{etc.,}$$

which we can easily prove is the case. Indeed, the condition

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}$$

means

$$\frac{\partial}{\partial y}\left(-f(r)\frac{x}{r}\right) = \frac{\partial}{\partial x}\left(-f(r)\frac{y}{r}\right),$$

or, since this is partial differentiation, where x is considered constant when deriving with respect to y; also y is considered constant when deriving as to x, we have simply

$$x\frac{\partial}{\partial y}\frac{f(r)}{r} = y\frac{\partial}{\partial x}\frac{f(r)}{r};$$

or, applying the rule of differentiating functions of functions we have

$$x\frac{d}{dr}\left(\frac{f(r)}{r}\right)\frac{\partial r}{\partial y} = y\frac{d}{dr}\left(\frac{f(r)}{r}\right)\frac{\partial r}{\partial x}.$$

But from $r^2 = x^2 + y^2 + z^2$ we have

$$\frac{\partial r}{\partial y} = \frac{y}{r} \qquad \text{and} \qquad \frac{\partial r}{\partial x} = \frac{x}{r},$$

hence our conditions become

$$\frac{xy}{r}\frac{d}{dr}\left(\frac{f(r)}{r}\right)=\frac{yx}{r}\frac{d}{dr}\left(\frac{f(r)}{r}\right),$$

an identity. The same would apply to the other two conditions. Hence U is the potential function sought.

Example. The force is such that its components upon the axes, x, y, z, are functions of the coordinates: $X = f_1(x)$; $Y = f_2(y)$; $Z = f_3(z)$. Find the potential function. Here

$$Xdx + Ydy + Zdz = d\left[\int f_1(x)dx + \int f_2(y)dy + \int f_3(z)dz\right] = dU;$$

Hence $U = \left[\int f_1(x)dx + \int f_2(y)dy + \int f_3(z)dz\right]$, provided that the criterion is satisfied.

We should not feel in any way compelled to use rectangular coordinates in compiling dW or dU. Thus, instead of

$$Xdx + Ydy + Zdz = dU$$

we might have $Pdp + Qdq + Rdr = dU$, if the position of the particle is defined by some special coordinates p, q, r. For instance in polar coordinates, r, θ, we have

$$Rdr + \Theta rd\theta = dU.$$

In semi-polar or cylindrical coordinates, r, θ, z, we have $Rdr + \Theta rd\theta + Zdz = dU$. It is easy to see why in both cases we have $\Theta rd\theta$ and not simply $\Theta d\theta$; because the latter (force times angular displacement) is *not work*. Now Θr is a moment and moment times angular displacement *is* work. So that the method of finding the potential function is always the same: find expression of elementary work and reduce it, if possible, to a complete differential of some function. If the criterion is satisfied, then the indefinite integral of such a function is U.

One more example will illustrate the mechanism of such reduction; it forms part of a problem given in Chapter V (see fig. 52).

An inspection of the working sketch of this system supplies the following expression for elementary work, corresponding to elementary displacement $d\varphi$

$$dW = -\lambda AMd(AM) - \mu BMd(BM) + Md(c \sin \varphi);$$

now this does not look like a complete differential of any function U; but, arranging it in the form

$$dW = Mc \cos \varphi d\varphi - \tfrac{1}{2}d(\lambda(AM)^2 + \mu(BM)^2),$$

also taking into account the general relations

$$(AM)^2 = a^2 + c^2 - 2ac \cos \varphi,$$
$$(BM)^2 = a^2 + c^2 + 2ac \cos \varphi,$$

we have

$$dW = c[a(\mu - \lambda) \sin \varphi + M \cos \varphi]d\varphi = dU;$$

hence U is the indefinite integral of dU

$$U = c[M \sin \varphi - a(\mu - \lambda) \cos \varphi]$$

(the constant does not matter for the present; later we shall be very specific in selecting it). It will be noted that in this case our only coordinate is angular, φ, and that the derivative of U as to φ gives us the force *corresponding* to that coordinate, that is a certain *moment*, not an ordinary force; because a *moment* multiplied by an angular displacement gives work. A *force* does not. The force understood in this broader sense is called *generalized* force, and in our future work we shall have a great deal to do with such forces.

Remembering that

$$\Sigma \frac{mv^2}{2} - \Sigma \frac{mv_0^2}{2} = U - U_0,$$

it will be seen that the integral of kinetic energy affords a very powerful principle for solution of problems of dynamics.

Example. A small material particle of mass m, slides, without friction, off a cylindrical log; to determine the point at which it leaves the log.

Instead of applying the fundamental equation, we can simply use the integral of kinetic energy. We know that the pressure exerted by a particle on a frictionless curve is equal to the centrifugal force, $m \cdot (v^2/r)$ *minus* the radial component of the applied force (Bowser, Anal. Mech., p. 348). It is clear that the particle will leave the log at a point where

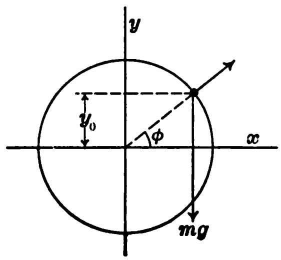


FIG. 7.

this pressure will vanish. The equation of the pressure is $P = m(v^2/r) - mg \cos \varphi$ (fig. 7). Now, when this is $= 0$, we have

$$\frac{v^2}{r} = g\frac{y}{r}$$

or $v^2 = gy$. Since the acting force is that of gravity, we have the potential function $U = -mgy$; so that

$$m\frac{v^2}{2} - m\frac{v_0^2}{2} = -mgy + mgy_0;$$

substituting $v^2 = gy$ we have

$$y = \frac{2gy_0 + v_0^2}{3g}.$$

If, at the beginning of motion, the velocity was 0, we would have had $y = \frac{2}{3}y_0$, that is the particle would have left the log after having descended one third its initial elevation above the center of the log.

It will thus be seen how a judicious introduction of the integral of kinetic energy simplifies the work; it is itself a partial solution of the problem. Another example will be given in the next article.

It is important to remember that the integral of kinetic energy was evolved from the fundamental equation, that is from the most general case that can be imagined, with the only limitation, that the constraints were independent of the time. Therefore the integral of kinetic energy can be applied to constrained as well as to free motion. This is evident from the fact that when we consider the work of *all forces* to be equal to the increase of kinetic energy, we imply the idea of the reactions being normal to the constraints and therefore producing or absorbing *no work* (we do not consider the effect of friction). So that in the case of constrained motion we also have

$$d\Sigma\frac{mv^2}{2} = \Sigma(Xdx + Ydy + Zdz),$$

if there is no potential function and

$$d\Sigma\frac{mv^2}{2} = dU,$$

if there is such a function, in which latter case we might also say that

$$\Sigma\frac{mv^2}{2} = U + h;$$

which is what we represented by T.

Synopsis of notations relative to work and energy. Virtual work $\delta W =$ force times displacement, $F\delta s$, that is force times any displacement projected upon the direction of the force, $= P\delta p \cos(P, \delta p)$. It also equals the sum of virtual works done by the components of the force upon their respective axes; that is, $= X\delta x + Y\delta y + Z\delta z$. In polar coordinates,

virtual work = moment $\times$ angular displacement, $M\delta\varphi$. The difference between δ and d is the arbitrary character of the former and the actual nature of the latter. D'Alembert's principle and the corresponding fundamental equation apply to any motion under any circumstances whatsoever, but can be simplified for special cases; thus for permanent constraints (in both shape and position), the formula (1) can be split into (2), of which the right side is elementary work, and the left side can be reduced to the form $d\Sigma m(v^2/2)$; this was briefly expressed thus: *kinetic energy = work done,* by which we mean that the increase of kinetic energy can only take place owing to external forces, and equals to the work done by them. We then agreed to denote the kinetic energy by T so that $dT = \Sigma(Xdx + Ydy + Zdz)$ is the same as (2) only in differential form. A still further reduction was made for the special case when $\Sigma(Xdx + Ydy + Zdz)$ could be shown to be an exact differential of some *potential function* U (of the coordinates only), that is, a function whose partial derivatives with regard to any coordinate equaled the forces corresponding to *that* axis. For this special case we had $dT = dU$ or

$$\tfrac{1}{2}d\Sigma mv^2 = dU; \quad \text{or} \quad T = U + h; \quad \text{or} \quad \Sigma\frac{mv^2}{2} = U + h;$$

each of which expressions is called the integral of kinetic energy. Such expressions exist only when the constraints are permanent and when, in addition to this, there is a potential function U; if there is no such function, there is no integral of kinetic energy but the general expression

$$dT = \Sigma(Xdx + Ydy + Zdz)$$

still holds true. In our future work we shall use various expressions of kinetic energy or of its integral in almost every problem; they should be well understood and memorized.

5. Integral of areas. This is another "short cut" principle, supplying a convenient method for beginning the investigation

of a problem; we shall give it in a form, slightly different from that adopted in Bowser's Anal. Mech., p. 487.

Let us write down the fundamental equation of dynamics for the case when a system can receive a virtual displacement *only* about the axis z; this will be an angular displacement, say $\delta\alpha$, and therefore it will be well to adopt polar coordinates, r, α, where r will be kept constant for each particle of the system (that is, no motion, except rotation about z). The usual formulae of substitution will be $x = r \cos \alpha$; $y = r \sin \alpha$; $z = 0$, we therefore have

$$\delta x = -r \sin \alpha \delta\alpha \qquad \text{and} \qquad \delta y = r \cos \alpha \delta\alpha,$$

or else, $\delta x = -y\delta\alpha$; $\delta y = x\delta\alpha$, and these values in the fundamental equation will give

$$\Sigma m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) = \Sigma(xY - yX),$$

which is usually written in the form,

$$\frac{d}{dt}\Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = \Sigma(xY - yX). \tag{1}$$

The right-hand member is well known from statics; it represents the sum of the moments of all forces about the axis z; the first member involves quantities $[m(dy/dt)]x$ and $[m(dx/dt)]y$; the expressions such as $m(dy/dt)$, etc., are known as the momentum of a moving particle projected upon the axis y; they are known from elementary courses (Bowser, Anal. Mech., p. 7) simply as mv, or mass times velocity. Each of these expressions, when multiplied say by x, is known as *moment of momentum*, or *angular momentum*, about the axis z (the axis which it does not contain); the same notations apply to the expression $[m(dx/dt)]y$; so that (1) can be expressed as follows: The time-derivative of the sum of the moments of momenta of all particles of a system about any axis, is equal to the sum of the moments of external forces about

that axis. The choice of the axes of coordinates is purely arbitrary and therefore we can select any line whatever as the axis about which the estimate of the moments of external forces is made; then the sum of angular momenta of the moving system about *that* axis, if differentiated with respect to time, will give the exact value of these external moments.

If, however, there are *no* external moments, the time-derivative is 0 and Σ is constant. Thus,

$$\Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = \text{const.} = a.$$

Substituting polar coordinates $x = r\cos\varphi$ and $y = r\sin\varphi$, we shall have

$$\Sigma mr^2\frac{d\varphi}{dt} = a$$

or, integrating,

$$\Sigma\int_{t_0}^{t} mr^2 d\varphi = a(t - t_0),$$

whereby the sum of areas described by the projected radii-vectors of each particle, each multiplied by the corresponding mass, is proportional to the time. This is the so-called integral of areas and is often used in either of the forms

$$\Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = a; \qquad \text{or} \qquad \Sigma mr^2\frac{d\varphi}{dt} = a. \tag{2}$$

Similar expressions obtain for the other axes. This integral can be used when there are no external forces acting on a body; or when the resultant of all forces passes through or is parallel to the axis about which it is desired to apply the integral of areas; in other words when there is no moment about that axis.

The following interesting problem will illustrate the application of this principle.

Example. A particle of unit-mass is constrained to move on the surface of a paraboloid of revolution $x^2 + y^2 = 2az$, *whose*

axis z is vertical. The initial velocity of the particle is horizontal. It is required to calculate the reaction of the surface in terms of z. Also to discuss the case when the initial velocity is $=\sqrt{2gz_0}$, where z_0 is the initial value of the coordinate of the particle, and finally to show that the particle will move in a circle in a horizontal plane and that the period of revolution will be the same for all such parallel circles.

The impressed forces reduce themselves merely to the weight of the particle: $X = 0$; $Y = 0$; $Z = -g$. The potential function is $U = -gz$. The integral of kinetic energy (m being $= 1$) is

$$\frac{v^2}{2} - \frac{v_0^2}{2} = U - U_0 = -g(z - z_0);$$

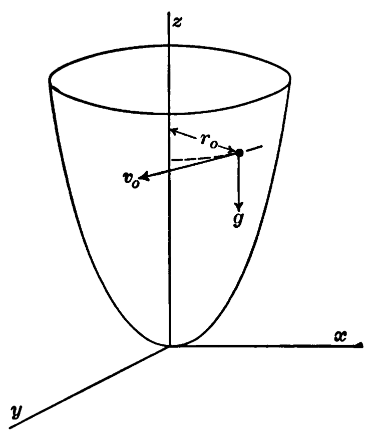
but from the general expression of velocity in rectangular coordinates

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2;$$

therefore

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 2g(z_0 - z) + v_0^2.$$

The surface being one of revolution about z, it is evident



that whatever the reaction is it must be normal to the surface, that is that it must pass through the axis of z, and therefore the reaction will give no moment about z. Neither will the impressed force (gravity), for it is parallel to the axis z (fig. 8). This is, therefore, a typical case where the integral of areas should be applied, viz.,

$$x\frac{dy}{dt} - y\frac{dx}{dt} = a;$$

where a is to be found from initial conditions. Let, for instance, the particle start from a position in the $x - z$ plane, so that $y_0 = 0$; we also have, the initial velocity being horizontal, $(dy_0/dt) = v_0$; therefore the integral of areas for these initial conditions reduces to $x_0v_0 = a = v_0r_0$, where r_0 will represent the initial distance of the particle from the z axis. Having thus found the constant a, we can write the general expression of the integral of areas as

$$x\frac{dy}{dt} - y\frac{dx}{dt} = v_0r_0.$$

Substituting polar coordinates ($x = r\cos\varphi$ and $y = r\sin\varphi$), we can transform the equation of the surface into $r^2 = 2az$; the integral of kinetic energy becomes

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2 + z^2 = 2g(z_0 - z) + v_0^2;$$

and the integral of areas will be

$$r^2\frac{d\varphi}{dt} = v_0r_0.$$

Differentiating the equation of the surface, $r^2 = 2az$, we have

$$r\frac{dr}{dt} = a\frac{dz}{dt};$$

substituting the values of $(d\varphi/dt)$ and (dr/dt), found from the

last two equations, into the integral of kinetic energy we have

$$\frac{a^2}{r^2}\left(\frac{dz}{dt}\right)^2 + \frac{v_0^2 r_0^2}{r^2} + \left(\frac{dz}{dt}\right)^2 = 2g(z_0 - z) + v_0^2.$$

In order to eliminate r_0^2 and r^2, we can substitute their value from the equation of the surface, that is $r_0^2 = 2az_0$, and $r^2 = 2az$, after which, solving for dz/dt, we have

$$\left(\frac{dz}{dt}\right)^2 = \frac{2z[2g(z_0 - z) + v_0^2] - 2v_0^2 z_0}{a + 2z}$$
$$= \frac{4g}{a + 2z}(z - z_0)\left(\frac{v_0^2}{2g} - z\right).$$

This is satisfied by putting

$$\frac{v_0^2}{2g} - z = 0;$$

indeed, in that case,

$$z = \frac{v_0^2}{2g} = \text{const.} \qquad \text{and} \qquad \frac{dz}{dt} = 0.$$

Now $z = \text{const.}$ means that the motion will be in a plane parallel to $x - y$. Substituting z just found in the equation of the surface we have

$$r^2 = \frac{av_0^2}{g};$$

whence the angular velocity

$$\omega = \frac{v_0}{r} = \sqrt{\frac{g}{a}};$$

and the period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{a}{g}};$$

in other words the period is independent of z. In this typical example it was possible to use the integral of kinetic energy as well as that of areas. In the example given under *D'Alembert's principle* (a particle in a plane rotating about a vertical axis) no such application could have been made; the moving

constraints (containing t) made it impossible to use the integral of kinetic energy (see how this was derived for the case of constraints involving no time); likewise, although the impressed force mg does not give any moment about z, the reaction of the plane *does* (being perpendicular to the plane), so that the integral of areas cannot be used either.

In problems of that sort it is necessary to apply the general methods, the fundamental equation, or at least D'Alembert's principle in its original form, as given above.

6. Coordinates of a rigid body. When we speak of a *system*, we mean two or more material particles, in some manner connected together. This connection will prevent the absolute freedom of the motion of each particle and will therefore constitute what we have characterized as *constraints*.

We shall principally have to do with *internal, material, kinematical* constraints, such as for instance a string by means of which the distance between two stones cannot exceed a certain value; or a car track; or, say, a rigid body, all the particles of which are subject to the condition that the original form of the body is permanent; etc. But some *dynamical* constraints can be imagined, acting in precisely the same manner as would kinematical constraints, but involving no material means of any kind; the solar system is an example of this sort, where the planets under the action of the so-called *forces* (the true nature of which we do not understand), describe paths just as definite as if they were running on tracks.

At any rate the idea of constraints necessarily implies our knowing *all*, or at least *something* of the relative position of the particles constituting the system. If we have n particles, of which the coordinates are, say, x, y, z, x_1, y_1, z_1, $\cdots$, etc., then any geometric relation of such form as

$$f(x, y, z, x_1, y_1, z_1, \cdots) = 0,$$

will mean some sort of a condition, in view of which the

particles are not altogether free to move; it is called, as we have seen before, an equation of constraint. We may have several such constraining equations $f_1 = 0$; $f_2 = 0, \cdots$, etc., except that their number cannot be equal to $3n$, the total number of coordinates. If such were the case, we could definitely establish, from these constraining equations, all the $3n$ coordinates of all the particles; in other words, the system would then be permanently fixed, and there could be no question of either equilibrium or motion. If the number of constraining equations is $3n - 1$, that is one less than the number of coordinates, then each particle can move only along one definite path, and moreover, such motion will cause perfectly definite displacements of all other particles of the system.

This would be the so-called system with *one degree of freedom;* its position can be fully determined by some one characteristic, for instance by one given point on any of the paths of the particles. Such characteristics, which are not necessarily coordinates in the usual sense, but may be distances from fixed points or angles (as will be shown later) are called *parameters.* If the constraining equations are $3n - 2$, that is *two* less than the number of coordinates, then each particle, instead of moving along a definite *curve,* will be able to move upon a certain surface, and two characteristics will be required (not necessarily coordinates in the old-fashioned sense, but distances or angles from certain fixed positions) to fully locate one of the particles on its surface; and that will fully determine the position of the system. In other words there will be *two degrees of freedom,* and *two parameters* necessary to characterize the system. In general, if the number of particles is n, and the number of limiting conditions (constraints) is k, then the number of parameters necessary to fix the position of the system will be $3n - k$; they may be either coordinates or other typical characteristics in the sense indicated above. They are to be determined from the equations of motion; so

that, in general, the problem will involve $6n - k$ variables to be determined: $3n$ coordinates of the n particles, and $3n - k$ parameters.

In order to do this in a definite way we must have the same number of equations; now we have available $3n - k$ equations of motion, also the like number of equations connecting the parameters with the coordinates of the particles, and the k equations of constraints, or $6n - k$ equations, in view of which the problem will be definite. By way of illustration, let us take a crude example of two particles given by their coordinates x_1, y_1, z_1, and x_2, y_2, z_2; the fact that they are two in number does not constitute a system. If we impose some condition, for instance, fixing the distance between the particles, which is expressed very easily by rules of elementary space geometry, then this connection will change *two separate particles* into a *system* of two particles; here we have 6 coordinates and one equation of constraint

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - l^2$$
$$= f_1(x_1, y_1, z_1, x_2, y_2, z_2) = 0 \quad (1)$$

and the system will thus have five degrees of freedom. That is to say one of the particles can have any motion whatsoever (three coordinates) and the other can be anywhere on a sphere, radius l, described about the first (two more coordinates, angular or any other). Taking the other extreme case suppose now that we have *five* constraining conditions, such as for instance $x_1 = a$; $y_1 = b$; $z_1 = c$; $y_2 = d$ and finally $f_1 = 0$, as (1). The first three equations permanently fix the first particle; the fourth equation, $y_2 = d$, means that the second particle can move only in a plane parallel to the plane $x - z$; and $f_1 = 0$, maintains the distance between the two particles equal to l. Therefore our system will be capable of moving in a manner shown in fig. 9; the apex of the cone will be the point x_1, y_1, z_1; the length of its generator will be l and the end of the latter, that is the movable particle x_2, y_2, z_2,

will be capable of assuming any position on the circle in the plane B, parallel to the plane $x - z$. Since we have five imposed conditions and six coordinates of the particles, the

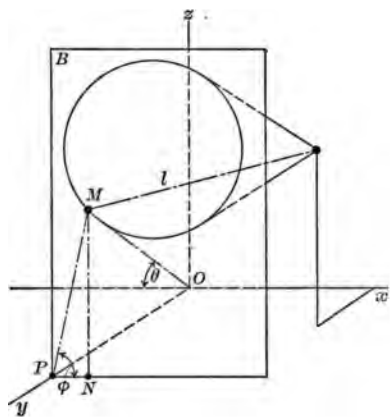


FIG. 9.

system will have one degree of freedom, that is to say, only one parameter will be required to definitely locate the whole system. This parameter may be selected in a great variety of ways; for instance we might take the distance MO; or the angle θ of the line MO with the y axis; or the coordinate MN; or PN, or PM, etc. Our problem will then be to express with the aid of the equations of motion, the parameter as a function of time, since this will determine the position of the system for any given time. Of course great care will be exercised in selecting the parameter, so that the labor involved in solving the equations of motion will be a minimum.

We shall now take another supposition, namely, that there are only four imposed conditions, that is that the system has only two degrees of freedom. Let, for instance, $x_1 = a$; $y_1 = b$; $y_2 = d$ and $f_1 = 0$ as before. The difference between this and the preceding case is that the apex of the cone itself will here be indeterminate, except that it will move on a line, parallel to the z axis; therefore it will be necessary to have two

separate parameters to characterize such a system. One of them may for instance be the same as before, say MP, and the other simply the missing coordinate z; or again MP together with PN can be chosen; or two angles φ and θ, etc.

A rigid body may be said to consist of a very great number of separate particles, and we might think therefore that three times as many coordinates might have to be determined in our problem in order to characterize the whole body in its motion. Yet, in view of the constraints (mutual action, maintaining the original shape of the body) only six parameters are necessary. What they are, is implied by the six equations of elementary statics: in order to have equilibrium (that is the number of parameters equal to the number of coordinates) we must have six conditions, three of which refer to the center of gravity of the body and the other three, to the angular equilibrium about the axes x, y and z. Any coordinate of any point of the body can be readily expressed through these six parameters. By way of reviewing the above it is useful to mention the following:

1. A free particle has three degrees of freedom, and three parameters (coordinates) are necessary to identify it.

2. A particle constrained to remain upon a given surface has two degrees of freedom.

3. A particle moving upon a curve has one degree of freedom.

4. A rigid body, if free, has six degrees of freedom.

5. A rigid body having one point fixed has three degrees of freedom (rotations about the three axes).

6. A rigid body rotating about a fixed axis has one degree of freedom, and only one (angular) parameter is required to completely determine the position of the body.

We shall return to this important subject presently and show how Lagrange has based his wonderful method upon this fruitful conception: the possibility to identify a system by means of some characteristic parameters, the number of which is less than the number of separate coordinates of the particles.

7. Motion of rigid body. Many extended treatises have been written upon this subject; in the following few pages we can only briefly mention a few main principles involved in the study of the kinetics of rigid bodies. The reader should carefully review his stock of knowledge already acquired from elementary treatises (for instance Bowser, Anal. Mech., Chapters VI, VII and VIII) in addition to which the following remarks can be made.

(*a*) MOMENTS OF INERTIA. We know that there exists, about every point of a rigid body, a *momental ellipsoid,* which is the locus of all points, such that for each point the expression $1/r^2$ (the reciprocal of the square of the corresponding radius vector) is equal to the moment of inertia about the axis coinciding with that radius vector. In every such ellipsoid there are three rectangular axes, called *principal axes of inertia,* about which the products of inertia vanish, so that the equation of the ellipsoid contains only the principal moments of inertia A, B and C

$$Ax_1^2 + By_1^2 + Cz_1^2 = 1.$$

Knowing A, B and C we can readily express the moments of inertia of the body about any line passing through the center of the ellipsoid (whether this is the center of gravity of the body or not) and given by its angles α, β, γ, with the principal axes. Such a moment of inertia will be

$$H = A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma.$$

In this case the corresponding radius of gyration can be shown to be

$$k^2 = a^2 \cos^2 \alpha + b^2 \cos^2 \beta + c^2 \cos^2 \gamma,$$

where a, b and c are the radii of gyration corresponding to A, B and C.

From this equation the following deductions can be made: (1) The axis of the greatest moment of inertia coincides with

the smallest axis of the ellipsoid and inversely. (2) If two of the principal moments of inertia are equal, say $A = B$, the ellipsoid is one of revolution about the third axis C; in this case all axes forming the same angle with C will have the same values of the moments of inertia; and all axes perpendicular to C will be principal axes; with reference to any of these the moment of inertia will $= A = B$. (3) In order that a line may be a principal axis of inertia relatively to any of its points, it is only necessary to have the products of inertia vanish, that is $\Sigma myz = 0$ and $\Sigma mxz = 0$ (if the proposed line be taken as the z axis). In this connection it might be well to caution the reader against thinking that a principal axis must necessarily pass through the center of gravity of the body; principal axes may have as an origin any point within the body; however a principal axis passing through the center of gravity possesses a curious property, viz.: it will remain the principal axis of any set of axes built on any of its points.

A second important theorem, is the well-known one that the moment of inertia of a body about any axis is equal to the moment of inertia about any axis through the center of gravity, parallel to the given axis, plus the mass of the body multiplied by the square of the distance between the two axes. Hence the fact that the moments of inertia about the axes passing through the center of gravity are the least of all parallel axes; also, that the moments of inertia about all parallel axes, equidistant from the center of gravity, are equal.

Combining the above we may easily find the moment of inertia about any axis whatever, no matter where located and how inclined, if the principal moments of inertia about the center of gravity are known.

(*b*) BODY ROTATING ABOUT A FIXED AXIS. The moment of inertia can now be introduced into the formula already known as the *integral of areas*

$$\Sigma m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) = \Sigma mr^2 \frac{d\varphi}{dt} = a.$$

Here the first part (as has already been mentioned) is known as the sum of the angular momenta (or moments of momentum); the sum $\Sigma mr^2(d\varphi/dt)$ consists of two factors, of which the first Σmr^2 is the moment of inertia (about the axis z), and the second $(d\varphi/dt)$ is the rate of change of the angular displacement, known as angular velocity. For a rigid body instead of the *sum* of angular momenta we might simply use the expression *angular momentum;* so that finally the angular momentum, H, of a body is equal to the product, $I\omega$, of the moment of inertia into the angular velocity (all taken about the same axis). This theorem is already known (Bowser, Anal. Mech., p. 454) and will be constantly used in our future work.

Another interesting theorem gives the kinetic energy of a rotating body as half the product of its moment of inertia into the square of its angular velocity: the velocity of a rotating particle is equal to the radius multiplied by its angular velocity $v = r\omega$; therefore

$$\Sigma \frac{mv^2}{2} = \Sigma \frac{m\omega^2 r^2}{2} = \frac{\omega^2}{2} \Sigma mr^2 = I \frac{\omega^2}{2}$$

which also may be written $= Mk^2(\omega^2/2)$ (where k is the radius of gyration).

Still another theorem of equally great importance follows immediately from the equation (see under *Integral of areas*)

$$\frac{d}{dt} \Sigma m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) = \Sigma(xY - yX).$$

We have just seen that $\Sigma m[x(dy/dt) - y(dx/dt)]$ is the angular momentum, H, of the body and $= I\omega$; therefore its time-derivative dH/dt will be

$$= \frac{d}{dt} I\omega = I \frac{d\omega}{dt},$$

in other words

$$= I \frac{d}{dt}\left(\frac{d\varphi}{dt}\right) = I \frac{d^2\varphi}{dt^2};$$

so that the moment of forces about the axis of rotation equals the moment of inertia times the angular acceleration. All these theorems have been given in the elementary course, but it is of great advantage to see how easily and quickly they can be derived either from the fundamental equation of dynamics or at least from one of the great principles, following therefrom. (*Remark.*—The last three theorems have been derived only for the case of a body moving about a fixed axis. That of a body moving about a fixed point will be considered presently.) As an example let us take a ring (of a rectangular cross-section) whose outer and inner radii are a and b; its moment of inertia about its central axis will be immediately found by an easy integration (Bowser, Anal. Mech., p. 448) and its moment of inertia about any diameter of the ring will be one *half of that* (Ibid. Remark on Polar moment of inertia, p. 436). It is often customary to denote, in a body of revolution, the equal moments of inertia by A and B, while the moment of inertia about the axis of revolution is generally referred to as C. Therefore for the ring in question the three moments of inertia $A = B$, and C are all known:

$$A = B = \tfrac{1}{4}m(a^2 + b^2), \qquad \text{and} \qquad C = \tfrac{1}{2}m(a^2 + b^2).$$

If we now spin the ring about its central axis, we will have something like a gyroscope. Let the angular velocity of the spin be ω; then, according to the formula $H = I\omega$ (in our case $= C\omega$), we can calculate the angular momentum H of the rotating body; also its kinetic energy $I(\omega^2/2)$ (in our case $= C(\omega^2/2)$); also the necessary spinning moment, which will bring the system from rest to the angular velocity (of spin) $= \omega$, in, say, t seconds, the acceleration being constant and equal, say, a. The following easy reasoning will be applied: since the acceleration is constant, multiplying it, that is

$$\frac{d^2\varphi}{dt^2} = a,$$

by dt and integrating we have

$$\frac{d\varphi}{dt} = at + k,$$

where k is a constant of integration, to be derived from initial conditions; in the beginning of motion, that is when $t = 0$, there is no rotation, that is

$$\frac{d\varphi}{dt} = 0;$$

hence $k = 0$, and finally

$$\frac{d\varphi}{dt} = at;$$

but $d\varphi/dt$ is our (given) angular velocity ω; therefore $\omega = at$, whence $a = \omega/t$ is the constant acceleration $d^2\varphi/dt^2$. According to this the spinning moment which can, with this constant acceleration, bring the ring from rest to the angular velocity ω in given time, must be $= I(d^2\varphi/dt^2)$ or, in our notation, $= C(\omega/t)$.

It is of advantage to represent the angular momentum about its axis of rotation as a vector, directed *along* this axis, and in such a manner that the right-hand rotation of the body will advance the end of the vector like a cork screw. With this understanding we shall now explain the meaning of the theorem: the time-derivative of the angular momentum about any axis is equal to the sum of external moments about that axis (see (1) under *Integral of Areas*) that is

$$\frac{dH}{dt} = M$$

about any axis whatever. Let us consider the same ring of the moment of inertia, C, the angular velocity (of spin) being $= \omega$, as before. The angular momentum H about the axis of rotation will be represented (fig. 10) by the vector ab; now, if the ring, together with its axis, be displaced into a

new position *ac* in some plane *E–F*, we can easily see that the projection of the vector on the axis *F* will be decreased by a small amount *m*; while the projection of the vector on the axis *E*, which was originally 0, will now be *n*; therefore *n/dt*

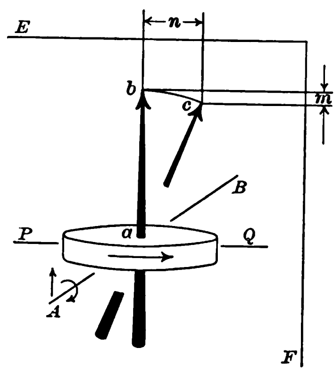


FIG. 10.

and *m/dt* will be the time-rates of change of the angular momentum-vector on these axes, that is, the time-derivatives of the angular momentum *about* these axes (taken at random).

Hence an elementary explanation of the action of a gyroscope (or top): a mere translation does not cause any particular effect because this does not alter the projection of the angular momentum upon any axis; but a rotation about an axis *A*, for instance, will immediately be felt by the observer: even a small rotation about *A* will change the projections of the angular momentum *upon* any axes such as *E* or *F*, and, as a consequence, external moments will arise, acting *about* these axes; the end *A* will have a tendency to rise, the end *B* to go down, precisely as if there were another moment applied about an axis *P–Q*, parallel to *E*.

(*c*) Body with one point fixed. (Consult Bowser, Anal. Mech., pp. 493–505.) In order to understand what follows the reader should thoroughly master the following remarks: (1) A rigid body having one point fixed can be displaced into

any desired position by means of a rotation about some axis, passing through the fixed point (Euler's theorem); the proof is left to the reader (Hint: Describe a sphere about the fixed point; mark two points corresponding to the initial position; also two points corresponding to the final position of the body, upon the sphere; try to find a general method by which the first two points can be displaced into their new position). In view of this theorem any motion whatever of a body about a fixed point may be regarded as a rotation about some continuously changing axis (we assume it changing, since if it were stationary, this case would degenerate into the one just considered under b), and with a definite (for that moment) angular velocity ω.

This axis is called the axis of *instantaneous rotation*, corresponding to the time t. It is well to note that the rotation, ω, itself is finite, although considered only during a very short period of time, after which it may change in both axis and magnitude. Now the rotation ω, being a quantity given in both magnitude and direction, is a vector, that is a directed quantity, and as such it may be represented, to a certain scale, by a portion of a straight line upon its axis; also it may be resolved into components, ω_1, ω_2, ω_3, upon any three fixed axes, such as x, y, z. These component rotations simply mean that three axes, fixed in a body, coinciding at the time t with three axes fixed in space, would *get away* from the latter, during the next element of time dt, by the corresponding amounts due to $\omega_1 dt$, $\omega_2 dt$, $\omega_3 dt$ (this is a so-called rotation of axes *about themselves*).

Suppose now that we have a point M (fig. 11), belonging to the body, and given by its coordinates x, y, z. It will tend to describe a circle about the instantaneous axis ω, although in reality it may describe only a very small arc thereof, owing to the motion of the axis itself. Nevertheless we can determine the linear components (upon x, y and z) of the velocity of the point M, during this very brief interval of time dt,

having at our disposal only these data: the coordinates x, y, z of the point and the corresponding components, ω_1, ω_2, ω_3,

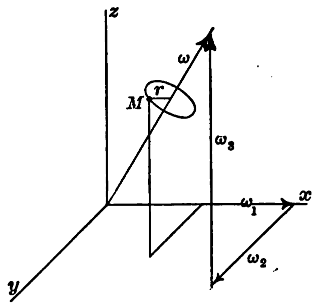


FIG. 11.

of the instantaneous rotation ω. These are given by *Euler's formulæ*

$$\frac{dx}{dt} = \omega_2 z - \omega_3 y,$$

$$\frac{dy}{dt} = \omega_3 x - \omega_1 z,$$

$$\frac{dz}{dt} = \omega_1 y - \omega_2 x.$$

The angles establishing the direction of the instantaneous axis by, or in terms of, its components ω_1, ω_2, ω_3 are found in elementary works on this subject; they are as follows: calling α, β, γ, the angles of the instantaneous axis with the fixed axes x, y, z, we have

$$\cos\alpha = \frac{\omega_1}{\omega}; \qquad \cos\beta = \frac{\omega_2}{\omega}; \qquad \cos\gamma = \frac{\omega_3}{\omega};$$

ω is evidently the resultant of its own components

$$\omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}.$$

This establishes the position of the axis with respect to any axes for which the components are given as ω_1, ω_2, ω_3). The

reader will note that the Euler's formulae can be solved for x, y and z, should this be required, *but these solutions would involve time-derivatives* (velocities). This is very important.

Having thus found the component velocities of any point of the body, we might derive a formula of the kinetic energy $\Sigma m(v^2/2)$ of the body for the same instant. We shall do it in a much shorter manner, however, remembering that the radius of gyration (see under *Motion of rigid body*) is given by

$$k^2 = a^2 \cos^2 \alpha + b^2 \cos^2 \beta + c^2 \cos^2 \gamma,$$

where a, b, c are the radii of gyration corresponding to principal axes of inertia A, B and C; so that

$$MK^2 = A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma.$$

On the other hand we know that the kinetic energy of a rotating body is given by $I(\omega^2/2)$, and furthermore that the cosines of α, β, γ, are

$$\cos \alpha = \frac{\omega_1}{\omega}; \qquad \cos \beta = \frac{\omega_2}{\omega}; \qquad \cos \gamma = \frac{\omega_3}{\omega};$$

so that, finally,

$$\frac{Mk^2\omega^2}{2} = T = \tfrac{1}{2}(A\omega_1{}^2 + B\omega_2{}^2 + C\omega_3{}^2)$$

which is generally written

$$2T = A\omega_1{}^2 + B\omega_2{}^2 + C\omega_3{}^2.$$

The reader will understand that heretofore the axes x, y, z were taken at random; but in this last calculation of T they were selected along the principal axes of the body, in order to do away with the products of inertia, which would appear for any other choice of axes.

Another very important vector will now be mentioned, which will probably be rather unfamiliar to the reader.

Under the *Integral of areas* we had the expression

$$\frac{d}{dt}\Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = \Sigma(xY - yX),$$

which was interpreted as follows: The time-derivative of the sum of the angular momenta is equal to the sum of moments of the external forces; by the sum of angular momenta (or, which is the same thing, moments of momenta) we mean the expressions similar to

$$\Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right).$$

Imagine, now, that we are considering a body moving in any manner about a fixed point; let us draw a set of principal axes of inertia through the fixed point, about which the moments of inertia will be A, B and C. The body, in general, will be rotating about some instantaneous axis and the components of rotation upon these axes will be say ω_1, ω_2, ω_3; by means of these it is easy to calculate the linear velocities dx/dt, dy/dt, dz/dt, according to Euler's formulae given above. Substituting these values of dx/dt, dy/dt, dz/dt, into

$$\Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$$

we have

$$\Sigma m\left(y\frac{dx}{dt} - x\frac{dy}{dt}\right) = \omega_3\Sigma m(x^2 + y^2) - \omega_1\Sigma mxz - \omega_2\Sigma myz;$$

but $\Sigma m(x^2 + y^2)$ is the moment of inertia, C, about the axis z, while Σmxz and Σmyz are products of inertia; they vanish because our axes are principal axes; we have, therefore,

$$\Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = C\omega_3$$

and two similar expressions

$$\Sigma m \left(y \frac{dz}{dt} - z \frac{dy}{dt} \right) = A\omega_1,$$

$$\Sigma m \left(z \frac{dx}{dt} - x \frac{dz}{dt} \right) = B\omega_2.$$

The vector, whose components upon the principal axes are the corresponding sums of angular momenta, that is $= C\omega_3$, etc., is called the *impulse-axis;* in our future work we shall use this vector rather freely; it is the *resultant* or *total angular momentum.* In other words we shall use the expression *impulse axis* precisely as an equivalent to *total angular momentum;* it will be denoted by P. To interpret the meaning of P graphically, let the point M (fig. 12), under conditions similar

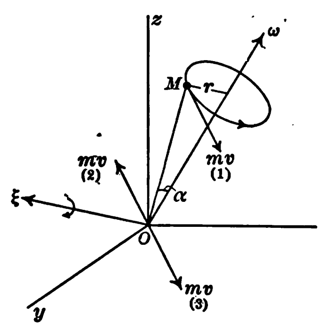


FIG. 12.

to those shown in fig. 11, be one of the points of the system, in temporary rotation, during a very short period, about the instantaneous axis ω. The momentum (1) due to this instantaneous rotation will of course be $= mv$, or, what is the same thing, $m\omega r$. If at the fixed point, O, we add two equal and opposite momenta, (2) and (3), we can readily see that the effect of the momentum (3) will vanish owing to the reaction of the fixed point, while the other two momenta, (1) and (2), will form a couple, of which the moment will be the

moment of momentum of the particle M in regard to the point O; this moment of momentum will be numerically $= mv \times OM$ (momentum times the arm) and can be represented (as would an ordinary couple) by a vector of a certain length, laid off upon a certain axis ξ; this axis must be: (1) perpendicular to the arm OM, and (2) must be in the plane $MO\omega$. Since $r = OM \sin \alpha$, it will also be equal to OM cos (ω, ξ), where by ω, ξ we mean the angle between these two axes. So that if the angular momenta are taken, about the point O, of all particles of the body, we can, adding such vectors as ξ, obtain their resultant which will be the total angular momentum about the fixed point O. This is precisely what we have just referred to as *impulse axis* and have denoted by P. Now if

$$\xi = mvOM = m\omega r \times OM = m\omega OM^2 \cos (\omega\xi),$$

we can also say that

$$\Sigma\xi \text{ (which is } = P) = \Sigma(m\omega OM^2 \cos (\omega\xi);$$

the projections of this resultant moment of momenta upon the principal axes are, as we have said before, $A\omega_1$, $B\omega_2$ and $C\omega_3$.

It is of interest to note that the kinetic energy, which we know to be $\frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2)$ is also

$$= I\frac{\omega^2}{2} = \frac{\Sigma mr^2\omega^2}{2} = \frac{\omega}{2}\Sigma mr^2\omega$$

$$= \frac{\omega}{2}\Sigma m\omega OM^2 \cos^2 (\omega\ \xi) = \frac{\omega}{2}P \cos (\omega\xi).$$

In other words the kinetic energy (body with fixed point only) is equal to the half-product of the total angular momentum (impulse axis) by the instantaneous angular velocity and by the cosine of the angle between their axes.

Taking again the general expression of kinetic energy

$2T = A\omega_1^2 + B\omega_2^2 + C\omega_3^2$, let us differentiate it partially, with respect to each of the variables ω_1, ω_2, ω_3, so that

$$2\frac{\partial T}{\partial \omega_1} = 2A\omega_1; \qquad 2\frac{\partial T}{\partial \omega_2} = 2B\omega_2; \qquad 2\frac{\partial T}{\partial \omega_3} = 2C\omega_3;$$

multiplying the first of these expressions by ω_1, the second by ω_2 and the third by ω_3, and adding the results we have

$$\omega_1\frac{\partial T}{\partial \omega_1} + \omega_2\frac{\partial T}{\partial \omega_2} + \omega_3\frac{\partial T}{\partial \omega_3} = A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = 2T,$$

which is another expression for the same kinetic energy. Comparing the coefficients of the like varables we have

$$A\omega_1 = \frac{\partial T}{\partial \omega_1}; \qquad B\omega_2 = \frac{\partial T}{\partial \omega_2}; \qquad C\omega_3 = \frac{\partial T}{\partial \omega_3};$$

that is, the projections of the total angular momentum upon any principal axis is equal to the partial derivative of the kinetic energy taken with respect to the velocity component upon that axis.

Remembering the very important theorem (see under *Integral of areas*) that the time rate of change of the total angular momentum about any axis is equal to the sum of external moments about that axis, we shall consider the causes to which such change of angular momentum may be due: there can be but two causes: (*a*) The vector itself, being not constant, but variable (with the time), may change by a small amount, so that its projection, say $A\omega_1$ at the instant t may vary by the amount $d(A\omega_1)$ in the next instant, so that the time rate of change will be $(d/dt)(A\omega_1)$; and (*b*) there is also rotation about the instantaneous axis, in which the whole body is taking part, including its principal axes as well as all vectors associated with them. We may express the time rate of change of the coordinates of any point x, y, z, upon any axes (otherwise called simply velocities); by Euler's formulae,

$$\frac{dx}{dt} = \omega_2 z - \omega_3 y,$$

etc.; so that the projection of the end of the vector P (which *is* the total angular momentum), will experience the effect of this double change in the very brief element of time, dt

$$\frac{d}{dt}(A\omega_1) + \omega_2 z - \omega_3 y.$$

This, then, is the total time-derivative of the total angular momentum, and as such, is equal to the corresponding moment of the external forces; in other words, calling the external momenta about the axes x, y, z, simply L, M, N (to simplify writing) we have

$$\frac{d}{dt}A\omega_1 + \omega_2 z - \omega_3 y = L;$$

and similar expressions for other axes, y and z. Now A is constant, so that the differentiation extends only to ω_1; and then, instead of any arbitrary point we will consider the end of the vector P, because it is *its* motion upon the axes that interests us, so that, substituting the coordinates of the end of P (that is, the projections of P itself upon the axes z and y which are: $B\omega_2$ and $C\omega_3$); we finally obtain

$$A\frac{d\omega_1}{dt} + (C - B)\omega_2\omega_3 = L;$$

the two other similar expressions for the other two axes are derived in the same manner:

$$B\frac{d\omega_2}{dt} + (A - C)\omega_3\omega_1 = M,$$

$$C\frac{d\omega_3}{dt} + (B - A)\omega_1\omega_2 = N.$$

These are the famous Euler's equations of the motion of a rigid body with one point fixed. They give, for every instant, the time variation of the angular velocity components ω_1, ω_2, ω_3, about the *principal axes*, in terms of the components

of external moments on *these* axes. In other words, if the external moments can be found about the principal axes, passing through the fixed point, we can find the rotation components ω_1, ω_2, ω_3 on these axes (and, therefore, the instantaneous axis); or, again, if the latter components are given, we can find the moments which are producing them.

Remembering that the *angular momentum* is the product of the moment of inertia by the angular velocity (see Bowser, Anal. Mech., p. 454) and that the components of the angular momentum upon the three principal axes are $A\omega_1$, $B\omega_2$, $C\omega_3$, we have the following pretty explanation of the meaning of Euler's equations (Bouasse, Mécanique, p. 563):

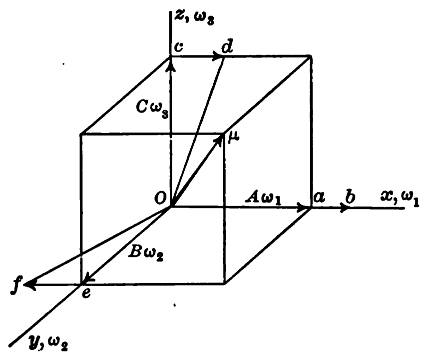


FIG. 12*a*.

The velocity of the tip of the vector μ, representing the angular momentum, is equal to the vector representing at that moment the applied couple (Fig. 12a).

Taking for example the first Euler's equation, let us find the velocity of μ as projected on the axis x. Such a velocity would consist of three parts:

1. $A(d\omega_1/dt)$; this would be the result of the change of the vector $A\omega_1$ with the time, and would be represented, say, by ab.

2. Owing to the rotation of the vector $C\omega_3$ under the action

of the component ω_2 there would be a second component $C\omega_2\omega_3$, represented, say, by *cd*.

3. Under the action of the component ω_3 the vector $B\omega_2$ would likewise contribute a component $B\omega_2\omega_3$, shown by *fc* and directed backward (since all rotations are clockwise, looking toward the center). Hence Euler's first equation,

$$A\frac{d\omega_1}{dt} + (C - B)\omega_2\omega_3 = L.$$

The other equations can be explained in the same manner.

The integration of these equations is rather difficult and we shall limit ourselves to the case when there are no external moments, that is when $L = M = N = 0$. This can happen only in the following cases: (*a*) When the particles of the system are under action of no forces; (*b*) when the particles of the system are subject to some attractive or repelling forces, emanating from the fixed point, O; (*c*) when the particles of a system are subject to mutual attraction according to any law, or (*d*) when the system is under the action of gravity and its center of gravity is at the fixed point, O.

Let us consider the last assumption, that is the case of gravity being the only external force and the center of gravity being at the center O. There will then be no external moments; that is $L = M = N = 0$; so that Euler's equations will be

$$A\frac{d\omega_1}{dt} + (C - B)\omega_2\omega_3 = 0$$

$$B\frac{d\omega_2}{dt} + (A - C)\omega_3\omega_1 = 0,$$

$$C\frac{d\omega_3}{dt} + (B - A)\omega_1\omega_2 = 0.$$

One of the first integrals is easily found by multiplying these equations respectively by ω_1, ω_2, ω_3, and adding

$$A\omega_1 d\omega_1 + B\omega_2 d\omega_2 + C\omega_3 d\omega_3 = 0$$

or, integrating,

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = \text{const.} = h.$$

But this $= 2T$, or the kinetic energy which is, therefore, constant as it should be since the outside forces perform no work. This is the *integral of kinetic energy.*

Again, multiplying the same equations by $A\omega_1$, $B\omega_2$, $C\omega_3$ and adding we have

$$A^2\omega_1 d\omega_1 + B^2\omega_2 d\omega_2 + C^2\omega_3 d\omega_3 = 0;$$

or, integrating,

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = \text{const.} = l^2.$$

Remembering that $A\omega_1$, $B\omega_2$, $C\omega_3$, are the projections of the resultant moment of momentum (or *impulse axis*) upon the principal axes, we see that l^2 is the square of the impulse axis itself, and that in this case it is constant, as it should be, since the outside moments $L = M = N = 0$; and the resultant angular momentum remains unchanged. The above expression is the *integral of areas* for our case; we shall return to Euler's equations when studying the motion of a rigid body.

8. Relative motion. It is useful to think of a coordinate system, in general, as of a mere temporary scaffolding, by reference to which motion can be identified by the observer or investigator; the axes possess no character of permanence and can be drawn in any convenient manner; moreover, in the most general case, the system, together with the axes to which it is referred, can have a bodily motion, translatory or rotational or both, in relation to some other set of axes. Therefore the observer, having determined the motion in relation to the original (so-called "reference") axes, but not having taken into consideration the motion of the reference axes themselves, would be surprised to see that the results of his calculations might be altogether different from what was actually observed. The earth, for instance, possesses a

number of very intricate motions in space, whose effect is hardly felt in our practical life; and yet a gyroscope or a pendulum can easily be made, which will actually reveal at least one of such motions, namely the rotation proper of the earth. In general, therefore, we shall consider two sets of axes (fig. 13); in the first place the *reference* axes x, y, z, to

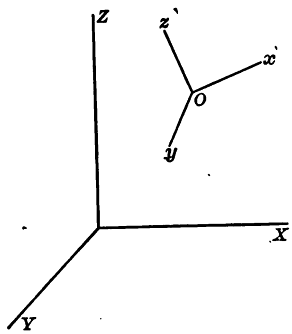


FIG. 13.

which the motion of a particle (or system) can be referred; and another set of *immovable* axes, X, Y, Z. The motion of the given system in relation (fig. 14) to the reference axes, x, y, z, is called *relative* motion; the motion of the reference axes themselves, in relation to the fixed, immovable set,

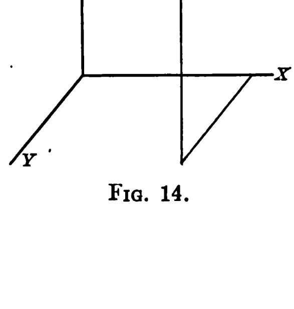


FIG. 14.

X, Y, Z, is called "*mouvement d' entraînement,*" for which there is no English equivalent; we shall refer to such motion as *MAR* (motion of axes of reference); and, finally, the result of these two motions would be the true motion of the system (or particle) in relation to the fixed axes, which is termed *absolute* motion. So that we have to deal with paths, velocities and accelerations, each of which may be (1) true, or absolute; (2) *MAR*, or of reference axes; and (3), relative, that is true only in relation to the relative axes, x, y, z. In this connection it is well to remember that, in general, relative equilibrium will mean absolute motion; absolute equilibrium will mean relative motion.

Very often the motion is such that the origins of both relative and absolute system of axes coincide; in this case we have merely rotation of one of the sets, x, y, z, in relation to the other X, Y, Z. In a more general case, however, the relative (or reference) system has both translatory and, at the same time, rotational motion in regard to the absolute system,

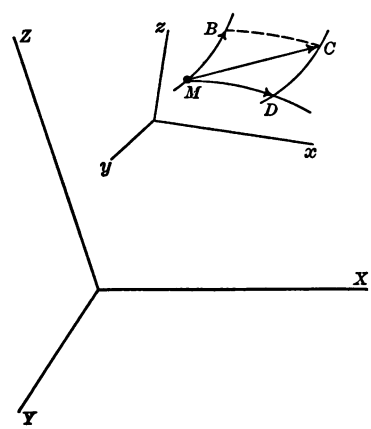


FIG. 15.

X, Y, Z. If, for instance (fig. 15) M is a particle, either taken by itself or belonging to some system, its relative path (that

is relative to the reference axes, x, y, z) may be such as M–B; at the same time the MAR motion of the axes proper would bring this same point into D; while the resulting, absolute, motion (that is referred to the fixed axes X, Y, Z, will result from these two motions and will be M–C. On the other hand another viewpoint can be taken: a particle M has absolute motion M–C; how will this motion appear to an observer, connected with some reference system x, y, z, which itself possesses a certain, known, motion in regard to X, Y, Z? The latter question is considered the real problem of relative motion, although both viewpoints are closely connected.

In view of what has just been said, the main problem of relative motion can be framed in the form of the following question: Assuming a certain system, referred to some axes, x, y, z, to be under the action of certain forces and also assuming that the motion of such a system can be determined in relation to these axes, according to the rules of dynamics, already established; will there be any change in that motion, if the axes themselves, together with the system, are moving in relation to another set of fixed axes, X, Y, Z? The answer is: the motion will be entirely different; it will be as if two new forces had been added to the already applied forces, and the motion *then* determined according to principles established above. The observer belonging to the original (reference) system, x, y, z, will record precisely this, corrected, motion; what these two additional forces are will appear presently.

We shall first establish the simple relations, connecting together the absolute and relative velocities, and also the absolute and relative accelerations. If, during a very short interval of time, Δt, a particle, in its relative, apparent, motion, moved from M to B, and, if, during the same period, the whole relative path moved, in its MAR motion, to a new position, 2, so that the particle M would be displaced from M to D, then the resulting position of the particle at the end

of the period Δt will be C, found from a parallelogram, built on these two displacements, relative, M–B, and MAR displacement, M–D; so that the distance actually passed by M will be M–C. In other words, speaking geometrically,

$$(MC) = (MB) + (MD),$$

where the brackets indicate that the addition will be not algebraic, but meant in the same sense as when adding forces according to the law of triangle of forces. Dividing by Δt and passing to the limit we have

$$\lim\left(\frac{MC}{\Delta t}\right) = \lim\left(\frac{MB}{\Delta t}\right) + \lim\left(\frac{MD}{\Delta t}\right)$$

but these limits all represent the time-rates of certain displacements, that is velocities of certain motions:

$$\lim\left(\frac{MC}{\Delta t}\right) = v_a = \text{absolute velocity;}$$

$$\lim\left(\frac{MB}{\Delta t}\right) = v_{mar} = \text{velocity of the reference axes;}$$

and

$$\lim \cdot \left(\frac{MD}{\Delta t}\right) = v_r, \text{ is the relative velocity;}$$

so that finally

$$(v_a) = (v_r) + (v)_{mar},$$

that is the absolute velocity is the sum of the relative and (mar) velocities. From this, of course, the relative velocity will be $(v_r) = (v_a) - (v)_{mar}$ all of which can well be represented by the triangle of velocities (fig. 16). Having this

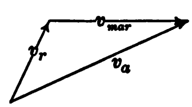


FIG. 16.

construction in mind, we can easily derive the analytical values of velocities by projecting them upon the desired directions (compare Bowser, Anal. Mech., p. 23).

The problem of finding the absolute acceleration in relative motion is much more intricate. It will be remembered that the usual definition of acceleration (fig. 17) is as follows:

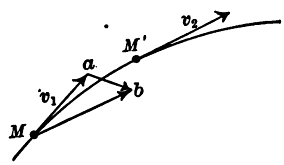


FIG. 17.

Let, at the time t, the velocity of a particle M be represented, in magnitude and in direction, by v_1; after a short interval of time, Δt, the new position of the particle will be M', and the new velocity will be, say, v_2; if we draw M–b equal and parallel to v_2, then a–b will represent the variation of velocity during the interval Δt; the limit of the ratio $ab/\Delta t$, of the variation of velocity to the corresponding time is what we call *acceleration of the particle corresponding to the time t.* (It is really the *velocity of velocity,* in other words the first time-derivative of velocity; or, the second time-derivative of space, dv/dt or d^2s/dt^2.) Its direction is along the acting force and its magnitude is F/M, the acting force divided by the mass of the particle.

To return to our problem of relative acceleration. Let (fig. 18) the relative path be (1) at the time t; let the bodily motion, MAR be of a translatory nature, so that after a short interval of time, Δt, the new position of the relative path will be (2) parallel to the former position (1); if at the time t the relative velocity is p and the (mar) velocity is r, then, after the interval Δt the new relative velocity will be, say, q, and the new (mar) velocity will be s; so that the corresponding accelerations will be, as explained above, a_r and a_{mar}.

Constructing an acceleration triangle, in precisely the same manner as we would a force triangle, we can say that under these conditions the resulting (absolute) acceleration

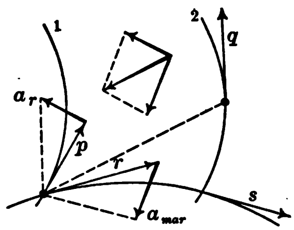


FIG. 18.

is the geometric sum of the two accelerations, of relative motion and of (*mar*) motion of the relative axes

$$(a_a) = (a_r) + (a_{mar})$$

(where the brackets are meant to show that the addition is geometric, not algebraic); this, indeed, is a simple rule, but it applies only to problems in which the (*mar*) motion is translatory; such problems are not general and not interesting; in general the (*mar*) motion is both translatory and rotational and for this reason a further investigation must be made.

Let the (*mar*) motion consist of a certain translation and of a certain rotation about some instantaneous axis (fig. 19);

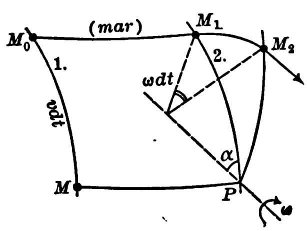


FIG. 19.

in other words let the relative path MM_0 be moved bodily into PM_1, and then receive a rotation in the direction ω about

the axis P–O, so that the final position of the path will be PM_2. A displacement of this double nature is the most general that can be imagined. (Owing to the very short period of time during which the displacement takes place, there can arise no question as to the order in which these displacements are to follow; the result will be the same whatever the order.) Knowing how to express the space traveled by means of acceleration and time (Bowser, Anal. Mech., p. 235: $s = \frac{1}{2}ft^2$) we can write $M_1M_2 = \frac{1}{2}a_c(dt)^2$ (where a_c is some sort of acceleration, producing a given displacement M_1M_2 in given time dt). On the other hand the radius of rotation can be expressed from the triangle POM_1; in other words $M_1O = v_rdt \sin \alpha$, and the small arc M_1M_2 will then be $M_1M_2 = v_rdt \sin \alpha \cdot \omega t$; comparing with M_1M_2, just found above, we have $a_c = 2v_r \cdot \omega \cdot \sin \alpha$. This is an entirely new acceleration (called *Coriolis's acceleration*) which gives us the desired correction. We can express it in the following easy manner: in order to find Coriolis's acceleration, project the relative velocity upon any plane perpendicular to the instantaneous axis of rotation and multiply by 2ω; finally turn through 90° in that plane, consistently with the direction of instantaneous rotation. The result will be the Coriolis's acceleration, which is also known as *compound centrifugal acceleration*. Indeed by projecting the relative velocity on any plane perpendicular to the rotation we have $v_r \sin \alpha$; by multiplying by 2ω we have the numerical value of a_c; and, since such acceleration is acting from M_1 toward M_2, we see that it must be perpendicular to both the axis of instantaneous rotation and to the relative velocity; hence—rotation through 90° in the plane perpendicular to P–O. This is the final solution of our problem: the acceleration of absolute motion is made up of accelerations of relative and (*mar*) motions and of the Coriolis's acceleration

$$(a_a) = (a_r) + (a_{mar}) + (a_c)$$

(where the brackets indicate the geometric nature of this addition). If the value of the *relative* acceleration is desired, we have, from the same equation

$$(a_r) = (a_a) - (a_{mar}) - (a_c);$$

or, since the absolute acceleration is equal to the resultant of forces actually applied, R, divided by the mass of the particle, we have

$$(a_a) = \frac{(R)}{m}.$$

Substituting in the above and multiplying by m,

$$m(a_r) = (R) - m(a_{mar}) - m(a_c);$$

But the last two terms also represent certain forces (each being a mass $\times$ acceleration); let us denote, therefore, the force $- m(a_{mar})$ by F_{mar} and the force $- m(a_c)$ by F_c; when we will have

$$m(a_r) = (R) + (F_{mar}) + (F_c) = \text{, say, } (R_1).$$

Hence

$$(a_r) = \frac{(R_1)}{m},$$

so that the relative acceleration is directed along the resulting force and is equal to the latter divided by the mass. In other words the relative acceleration can be found precisely as if there were no (*mar*) motion of relative axes; except that the force is not merely the applied force, R, but consists of the latter *plus two other forces,* one of which equals minus mass times acceleration, due to motion of relative axes ($- m(a_{mar})$) and the other of which equals minus mass times Coriolis's acceleration ($- m(a_c)$). These are the two additional forces of which mention has been made in the beginning of the problem; introducing them we can altogether neglect the relative nature of the motion and apply our ordinary principles. The motion thus found will be precisely such as may be ob-

served by an investigator, connected with the moving axes of reference.

We have indicated by brackets the geometric nature of the additions but it is equally easy to put the matter in an analytic form. Let X_0, Y_0, Z_0, be the components of R, the resultant of the forces actually applied, upon the reference axes, x, y, z; also let X_1, Y_1, Z_1, be the projections of the additional force (F_{mar}) and X_2, Y_2, Z_2, those of the other additional force (F_c) upon the same reference axes. Then we have merely three relations, one for each of the axes x, y and z, each expressing that mass times acceleration equals the sum of all forces upon that axis. That is, instead of

$$m(a_r) = (R) + (F_{mar}) + (F_c)$$

we have

$$m\frac{d^2x}{dt^2} = X_0 + X_1 + X_2;$$

and two other equations for the other axes. Integrating them we have six constants, which are supplied by the initial data of the problem.

These two additional forces (F_{mar}) and (F_c) will vanish if the (mar) motion of the axes of reference is uniform, rectilinear and translatory, as in this case both accelerations (a_{mar}) and (a_c) are 0; the force (F_c) will vanish if there is no relative motion $(v_r = 0)$; if the instantaneous axis is parallel to the relative velocity, $(\alpha = 0)$; or if $\omega = 0$; that is if the (mar) motion of the axes x, y, z, is translatory (as we have already seen). These deductions can readily be made from the formula $a_c = 2v_r\omega \sin \alpha$.

The following typical example will show the application of the above: *A material particle of mass m* (fig. 20) *is moving inside of a tube inclined at an angle α to a vertical axis, V, and in uniform rotation, ω, about the latter. It is required to determine the motion.*

Here the relative motion is that of the particle inside the tube, as viewed relatively to the tube; the (*mar*) motion reduces itself merely to the rotation of the tube (which is the relative path) about the vertical axis. Let s be the distance of the particle, at the time t, from the intersection with the

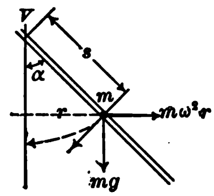


FIG. 20.

axis. The acting force is the weight mg; one of the two additional accelerations to be found is minus the (*mar*) acceleration; since this (*mar*) motion in our case means simply rotation, we have that its acceleration is ω^2r (centripetal force) and this is directed *toward* the axis of rotation. The first additional force will thus be $m\omega^2r$ and directed *away* from the center; the second (Coriolis's) additional acceleration acts in the following manner: it is located in the plane perpendicular to the axis; is tangent to the circle described by the radius r, and is directed forward, in the sense of rotation; therefore the force (F_c), which is *minus* mass times (a_c), is also tangent to the circle but directed backward. They are both perpendicular to the relative path (tube) and therefore do not affect the motion (no friction being considered) in any manner; the latter force, however, (F_c), represents the action of the particle upon the tube. In order to determine the motion along the tube the most natural way would be to project all forces upon its direction; having taken care of the additional forces, we can consider the relative motion (inside the tube) as absolute and apply our ordinary methods. The acceleration along the direction of the tube will be d^2s/dt^2, and the projected forces will be $mg \cos \alpha$ and $m\omega^2r \sin \alpha$ or $m\omega^2s \sin \alpha$

(since $r = s \sin \alpha$); therefore

$$m \frac{d^2s}{dt^2} = mg \cos \alpha + m\omega^2 s \sin^2 \alpha,$$

whence

$$\frac{d^2s}{dt^2} = g \cos \alpha + \omega^2 s \sin \alpha.$$

This is the final equation; in order to integrate it we first put it in the form

$$\frac{d^2}{dt^2}\left(s + \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha} \right) = \omega^2 \sin^2 \alpha \left(s + \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha} \right),$$

whence

$$s + \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha} = Ae^{\omega t \sin \alpha} + Be^{-\omega t \sin \alpha},$$

which gives the distance in terms of time; and, differentiating,

$$\frac{ds}{dt} = \omega \sin \alpha (Ae^{\omega t \sin \alpha} - Be^{-\omega t \sin \alpha}),$$

which gives the relative velocity. These equations contain two constants of integration, A and B, which will be determined from initial conditions. If, for instance, at the beginning of motion (when $t = O$) the particle is at V and has no initial velocity, we have

$$s_0 = 0; \qquad \frac{ds}{dt} = 0$$

whence

$$A = B = \frac{g \cos \alpha}{2\omega^2 \sin^2 \alpha};$$

substituting into the expressions for s and ds/dt just found, we have all that is necessary to specify the motion. Should the projection be required, on the horizontal plane, of the path of the particle in its absolute motion, we have but to

note that the polar coordinates of the particle will be

$$r = s \sin \alpha \qquad \text{and} \qquad \theta = \omega t,$$

which in the final equation for s will give a spiral.

A most important expression will now be established for the *absolute velocity* in relative motion. We have seen that this is made up, geometrically, of the relative velocity and of the (*mar*) velocity, of the motion of the reference axes themselves. An analytical expression will be derived as follows: Since any motion whatever can be decomposed into translation plus rotation about a certain instantaneous axis, let us take for origin (fig. 21) any point O in the moving body and let us

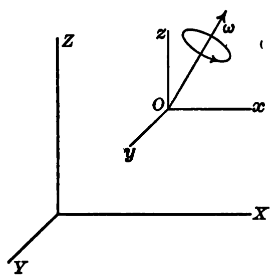


FIG. 21.

imagine the actual, absolute, motion resolved into two motions: the (*mar*) motion of the axes x, y, z (together with the body), which is to be purely translatory; and the relative motion of the body about some instantaneous axis ω. The latter motion will be the same as of a body with a fixed point (see under *Body with one point fixed*), since we have fixed the coordinates, x, y, z, in the point O of the body, so that there can be no other relative motion but rotation about some axis (ω) through this point. Since the (*mar*) motion is translatory, the axes x, y, z, being at all times parallel to the fixed axes X, Y, Z, it is evident that the (*mar*) velocity of all points of the body will be the same, and will be equal to that, say,

of the origin, O. Analytically therefore the projections of the absolute velocity on fixed axes, X, Y, Z, will be made up, (1) of the corresponding projections of (*mar*) motion, and these which are the same as projected velocities of *any* point O will be designated by v_{marx}, v_{mary}, v_{marz}; and, (2) of the corresponding projections (on the axes x, y, z, which are parallel to the fixed axes) of the velocities of the particle we are considering, due to the rotation of the system about the fixed point O. From the theory of rotation of the body whose one point is fixed the values of these velocities are (Euler's formulae)

$$\frac{dx}{dt} = \omega_2 z - \omega_3 y; \text{ etc.,}$$

so that finally the absolute velocities on fixed axes, X, Y, Z, are

$$\begin{aligned} v_{ax} &= v_{marx} + \omega_2 z - \omega_3 y, \\ v_{ay} &= v_{mary} + \omega_3 x - \omega_1 z, \\ v_{az} &= v_{marz} + \omega_1 y - \omega_2 x. \end{aligned} \tag{1}$$

These equations give the absolute velocities of a point, located by its coordinates, within the moving (reference) system x, y, z, of which the motion is the same as that of a body with one point fixed.

But we can imagine a still more general case, where the point itself will have a motion of its own within the moving system; let the velocity of this point on the moving axes be dx/dt, dy/dt, dz/dt. Since the velocities of this same point, stationary within the moving system, and moving *with* the latter, are given by (1), we shall now have

$$\begin{aligned} V_{ax} &= \frac{dx}{dt} + v_{marx} + \omega_2 z - \omega_3 y, \\ V_{ay} &= \frac{dy}{dt} + v_{mary} + \omega_3 x - \omega_1 z, \\ V_{az} &= \frac{dz}{dt} + v_{marz} + \omega_1 y - \omega_2 x, \end{aligned} \tag{2}$$

as projections of the absolute velocity of a particle, moving within a moving system of reference, a very broad expression, which we shall need in our future work.

To return to the equations (1), which are of exceedingly great importance, we can write them in the condensed form

$$v_{ax} = v_{marx} + \frac{dx}{dt}. \tag{3}$$

Let us suppose that instead of the arbitrary point O, we have chosen, as the origin of moving axes, the center of gravity, G. Let us, under this supposition, write down three equations similar to (3) for each point of the system, square them, multiply each by the corresponding mass and add the results together; this will give

$$\begin{aligned}\Sigma m(v_{ax}^2 + v_{ay}^2 + v_{az}^2) &= \Sigma m(v_{marx}^2 + v_{mary}^2 + v_{marz}^2)\\ &+ \Sigma m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]\\ &+ 2\Sigma m\left(v_{marx}\frac{dx}{dt} + v_{mary}\frac{dy}{dt} + v_{marz}\frac{dz}{dt}\right).\end{aligned} \tag{4}$$

Now the first term $\Sigma m(v_{ax}^2 + v_{ay}^2 + v_{az}^2)$ is evidently $= \Sigma m v_a^2$, v_a being the absolute velocity of the center of gravity; likewise the term $\Sigma m(v_{marx}^2 + v_{mary}^2 + v_{marz}^2) = \Sigma m v_{marg}^2$, or, simply, $M v_{marg}^2$ where v_{marg} is the total (*mar*) velocity of the motion of the center of gravity in its translatory movement; also

$$\Sigma m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right) + \left(\frac{dz}{dt}\right)^2\right] = \Sigma m v_r^2$$

because dx/dt, dy/dt, dz/dt are merely projections of the velocity on the reference axes x, y, z (in our case projections of the relative velocity v_r). The last term can be rearranged as follows: Omitting the brackets we have three terms

$$2\Sigma m v_{marx}\frac{dx}{dt}, \qquad 2\Sigma m v_{mary}\frac{dy}{dt}, \qquad 2\Sigma m v_{marz}\frac{dz}{dt};$$

but, since the motion is translatory, v_{marx} is the same for all particles, including the center of gravity, G (and the same applies to v_{mary} and v_{marz}); so that these velocities can be placed before the Σ sign:

$$2v_{marx}\Sigma m\frac{dx}{dt}, \qquad 2v_{mary}\Sigma m\frac{dy}{dt}, \qquad 2v_{marz}\Sigma m\frac{dz}{dt}.$$

But, for the center of gravity, $\Sigma mx = 0$; $\Sigma my = 0$; $\Sigma mz = 0$; (Bowser, Anal. Mech., p. 109) and therefore also

$$\Sigma m\frac{dx}{dt} = 0; \qquad \Sigma m\frac{dy}{dt} = 0; \qquad \Sigma m\frac{dz}{dt} = 0.$$

The whole last term of (4) is therefore 0, and we have

$$\Sigma m(v_a)^2 = Mv_{marg}^2 + \Sigma mv_r^2,$$

or, dividing by 2,

$$\Sigma\frac{m(v_a)^2}{2} = \frac{Mv_{marg}^2}{2} + \Sigma\frac{mv_r^2}{2}.$$

This theorem (called Koenig's theorem) is of greatest importance in practice and will be constantly used in our future work; it tells us that the total kinetic energy of any system is equal to the kinetic energy of its center of gravity (with the whole mass concentrated in it) plus the kinetic energy of the whole system, in its relative motion about its center of gravity (considered as fixed). *Caution.*—In calculating the kinetic energy of the center of gravity, $mv^2/2$, absolute, not relative velocity should be used; it is geometrically compounded of relative and (*mar*) velocities.

Let us take a few examples.

1. *A thin horizontal rod, of the mass m, can slide in a holder, which can turn about a vertical axis. The holder has negligible mass and there is no friction. Find the general expression of kinetic energy in terms of the angular velocity of rotation* (fig. 22).

Let C be the middle of the rod (the center of gravity).

According to Koenig's theorem we have to calculate, (1) the kinetic energy of the middle point, C (the whole mass being

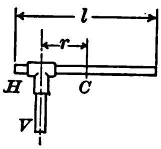


FIG. 22.

assumed concentrated in it), in its rotation about the vertical. This of course

$$= \frac{mv^2}{2} \neq \frac{m}{2}\frac{d^2s}{dt^2} = \frac{m}{2}\,\frac{dr^2 + r^2 d\varphi^2}{dt^2}$$

(compare Bowser, Anal. Mech., p. 243, also, Bowser, Calculus, p. 178). To this we must add (2) the kinetic energy due to the motion of the rod about its center of gravity, in other words, to its rotation, of which the instantaneous value will be the same as $d\varphi/dt$, the rotation about the vertical support. (This is often misunderstood: a well-known theorem of kinematics tells us that the axis of rotation can be transported (parallel to itself) to any point, provided that a certain translatory motion is added; but the angles of rotation will be precisely the same for all such parallel axes; hence, instead of rotation about V we may simply consider that about C. This remark will apply to almost every problem of this sort; we shall transfer centers of rotation freely, into any desired position.)

Now, the kinetic energy of a rotating rod $= \frac{1}{2}$ moment of inertia $\times$ the square of angular velocity, that is $(m/24)l^2(d\varphi/dt)^2$ (Bowser, Anal. Mech., p. 431); so that the total kinetic energy will be

$$T = \frac{m}{2}\left[\left(\frac{dr}{dt}\right)^2 + \left(r^2 + \frac{l^2}{12}\right)\left(\frac{d\varphi}{dt}\right)^2\right].$$

We note that this result is in terms of variables dr/dt, r and

$d\varphi/dt$, of which we know nothing, so far; it will be shown in the next chapter how, from expressions of T, found in the general form, similar to the expression just calculated, the equations of motion can be easily derived by Lagrange's beautiful method.

2. *Find the kinetic energy of a ladder (mass m) the top of which is resting on a smooth lamp-post, while the bottom is slipping, in some manner, on the icy sidewalk (no friction).* (By saying that the ladder is slipping "in some way" we mean to say that the motion of the bottom of the ladder will not be necessarily radial, away from the post, but may have a rotation, at the same time, about the post, as shown by the angle θ (fig. 23).)

Here according to Koenig's theorem we have: (1) the center C of the ladder is moving on a sphere, described from O with

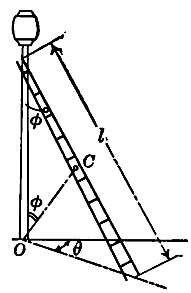


FIG. 23.

the radius $= l/2$ (check this statement by elementary methods of Anal. Geometry); therefore the velocity of C can be made up of very small displacement-velocities: along the meridian, $l/2 \cdot d\varphi/dt$ and along the arc of a parallel circle, $(l/2) \sin \varphi$ $(d\theta/dt)$; hence the expression

$$m\frac{l^2}{8}\left[\left(\frac{d\varphi}{dt}\right)^2 + \sin^2 \varphi\left(\frac{d\theta}{dt}\right)^2\right].$$

Now the relative motion of the ladder, that is to say, about

its center as fixed point, may be about a horizontal axis, if sliding outwardly, and, also, about a vertical axis, in its rotation about the post; so that we have to consider both rotations. Taking the ladder merely as a rod of the same mass and length, we have its moment of inertia = $ml^2/12$, therefore this part of kinetic energy will be

$$\frac{ml^2}{24}\left[\left(\frac{d\varphi}{dt}\right)^2 + \sin^2\varphi\left(\frac{d\theta}{dt}\right)^2\right].$$

(Note that $(d\theta/dt)\sin\varphi$ indicates that the rotation about the post, $d\theta/dt$, has been resolved, along, and at right angles to the ladder; and only the latter component taken into consideration; draw a sketch, directing $d\theta/dt$ vertically along the lamp-post and illustrate this remark.)

Finally we have

$$T = \frac{1}{6}\left[\left(\frac{d\varphi}{dt}\right)^2 + \sin^2\varphi\left(\frac{d\theta}{dt}\right)^2\right] mC^2$$

(*Remark.*—Instead of $T = \Sigma(mv^2/2)$ it is customary to write $2T = \Sigma mv^2$, and we shall make frequent use of this notation; likewise, in dealing with rods, as was the case in the last two examples, the length is generally denoted $2a$ or $2l$ to simplify writing.)

3. *A rod (mass m, length 2l) can slide, without friction, on a horizontal plane; its end A* (fig. 24) *cannot leave the axis x,*

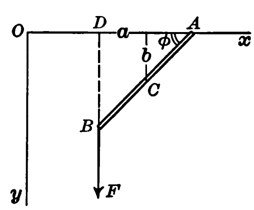


FIG. 24.

while its end B is acted upon by a repelling force F, perpendicular to O–x and proportional to BD, the ordinate of B. Find 2T in terms of φ.

This is a somewhat more intricate case. Let a and b be the coordinates of the center of gravity of the rod. The only forces applied to the rod consist of F and of corresponding reaction of the axis x; they are both perpendicular to the axis x, hence, with regard to this axis, the equation of motion can be written

$$m\frac{d^2a}{dt^2} = 0, \qquad \text{or} \qquad a = kt + n;$$

where k and n are the constants of integration. This means that the motion of the center of gravity, parallel to the axis x, will be uniform. The general expression of the velocity of C will be

$$V^2 = \left(\frac{da}{dt}\right)^2 + \left(\frac{db}{dt}\right)^2 = k^2 + l^2\cos^2\varphi\left(\frac{d\varphi}{dt}\right)^2,$$

$$(\text{since } b = l\sin\varphi),$$

which gives the velocity of the instantaneous bodily motion of the rod. We now must calculate the kinetic energy due to the rotation of the rod about its center with the angular velocity $d\varphi/dt$, which will be $\frac{1}{2}(ml^2/3)(d\varphi/dt)^2$ and the total kinetic energy will be

$$2T = mV^2 + \frac{ml^2}{3}\left(\frac{d\varphi}{dt}\right)^2 = m\left[k^2 + l^2(\tfrac{1}{3} + \cos^2\varphi)\left(\frac{d\varphi}{dt}\right)^2\right].$$

9. Euler's angles. We shall terminate this chapter by deriving Euler's angles, which will be often used in our work. We have seen (under *Body with one point fixed*) that the motion of a body about a fixed point resolves itself simply into a rotation, with a certain angular velocity, ω, about some instantaneous axis, whose projections upon any rectangular axes, ω_1, ω_2, ω_3, are connected together by the relation

$$\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2.$$

These projected rotations give us an idea as to the rotary displacement, which is going to take place, during the next

instant, about each axis; but they do not help us directly to characterize or identify the position of the body at a given time. Indeed, in order to do this we would have (1) to imagine a system of axes, x, y, z, say, the principal axes, fixed in the body, that is to say, moving with it; and (2) a system of axes X, Y, Z, fixed in space. (3) We would then have to find the cosines of each axis x, y, z, with each of the fixed axes, X, Y, Z (by rules of analytic geometry), that is nine cosines for the whole set of movable axes x, y, z, and, finally, (4) we would derive expressions of time-derivatives of all these cosines which would enable us to watch the motion of the axes x, y, z, that is of the body, in relation to a fixed set of axes, X, Y, Z. All this can be done, in a much simpler way, by Euler's angles and the formulae that can be derived therefrom.

According to Euler's method, in order to change the position of axes from any initial position, X, Y, Z, to any current position, x, y, z (fig. 25) only three rotations are necessary:

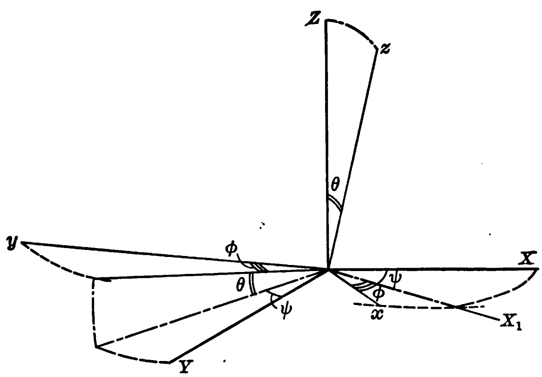


FIG. 25.

(1) about Z, through an angle ψ; (2) about the new position X_1–O, through an angle θ; and (3) about the new axis, z–O, through an angle φ. This completely identifies the new position of the axes, and therefore, the new position of the system; therefore, if we could watch the motion of the system by means

of the components ω_1, ω_2, ω_3, which, through the change of the nine direction-cosines could tell us where the axes would be during the next instant, we certainly could do so, with much greater ease, by following the changes of Euler's angles of which we have but three. Indeed, the resultant of the three small variations of Euler's angles will give the body the same instantaneous twist as would result from the three component rotations ω_1, ω_2, ω_3. Our problem is then to replace ω_1, ω_2 and ω_3 by expressions involving only the angles ψ, θ and φ, as well as their derivatives (angular velocities), $d\psi/dt$, $d\theta/dt$ and $d\varphi/dt$; in other words to follow the motion of the (moving) principal axes, along which it is customary to direct the moving axes x, y, z, *entirely through the angles* ψ, θ and φ, reckoned from some fixed system of axes X, Y, Z.

Now, since the instantaneous rotation ω can be resolved into any desired components, let us resolve it into three components, along the axes O–Z, O–X_1 and O–z (fig. 26). This

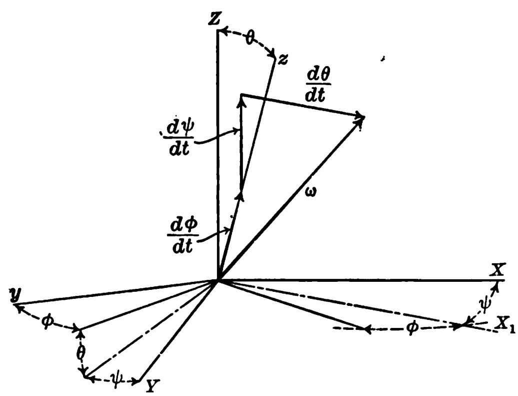


FIG. 26.

means that, instead of old components of instantaneous rotation, ω_1, ω_2, ω_3, we shall have the following new components (which, combined together, will give, of course, the same effect, ω): About O–Z $\cdots$ $d\psi/dt$, about O–X_1 $\cdots$ $d\theta/dt$, about

$O\text{–}z \cdots d\varphi/dt$. All we have now to do is to project $d\psi/dt$, $d\theta/dt$ and $d\varphi/dt$ upon the *moving* axes x, y, z (remembering that the projection of ω is equal to the sum of projections of its components). This gives

$$\omega_1 = \frac{d\psi}{dt}\sin\theta\sin\varphi + \frac{d\theta}{dt}\cos\varphi,$$

$$\omega_2 = \frac{d\psi}{dt}\sin\theta\cos\varphi - \frac{d\theta}{dt}\sin\varphi,$$

$$\omega_3 = \frac{d\psi}{dt}\cos\theta + \frac{d\varphi}{dt}.$$

It is very advisable for the reader to derive these formulae several times; the mechanism of projecting the components is clearly indicated on the drawing; in practice, as has been said before, for axes x, y, z, we generally take the principal axes of the body, which is thereby fully identified in reference to some fixed axes, such as X, Y, Z. It is well to remember that Euler's angles, ψ, θ and φ are positive in the direction indicated by the arrows; rotations in an opposite direction would be assumed negative. The reader will probably see at once why the elementary rotations $d\psi/dt$, $d\theta/dt$ and $d\varphi/dt$ have been projected upon the moving axes (or, rather, upon directions, coinciding at that instant, with the moving axes, x, y, z), while it would be just as easy to project them upon the fixed axes, X, Y, Z; the real advantage is this: by projecting the elementary rotations upon the moving axes we have means of locating, for any instant, the position of the instantaneous axis in relation to these moving axes; while the moving axes themselves are located at any instant, in respect to the fixed axes, X, Y, Z, by Euler's angles ψ, θ and φ. In other words, as soon as Euler's angles are given (in terms of the time), the reader will picture to himself two things: (1) by means of these angles, the position, for any time, t, of the moving axes (and therefore that of the body itself), can

be instantly established; and (2) within these moving axes, the instantaneous axis of rotation, can be established by its three projections ω_1, ω_2 and ω_3 on these same axes, which, furthermore are given in terms of these same Euler's angles. The latter are, therefore, sufficient to completely identify the position of the body at any given moment of time. The reader will no doubt understand that in problems involving spinning bodies (tops, gyroscopes, etc.) the axis of spin and the instantaneous axis ω are two entirely different things; in using Euler's angles for such problems it is customary to direct the Euler's axis z along the axis of spin, in which case the angular velocity $d\varphi/dt$ is the constant velocity of spin; while the instantaneous axis ω may be entirely outside of the rotating body; these two axes can coincide only in a very few problems, involving, for instance, spinning of a top about its steady vertical position.

By way of illustration we shall take the following very simple exercise, although it is of extremely great importance: *Supposing that a point is given in reference to a new system, x, y, z, by its coordinates, x, y and z; and that this new system is characterized by Euler's angles, ψ, θ and φ in relation to the old (or fixed) system, X, Y, Z; how can the position of the point be found in direct reference to the old coordinate system?* In other words, how to find X, Y, Z, in terms of x, y, z, and of the angles, ψ, θ and φ? In order to find these we have only to find expressions connecting together the two sets of coordinates and then to solve them for X, Y, Z. The reader is advised to draw two sets of axes to a large scale and then to proceed as follows: the new system was obtained from the old one by three successive rotations ψ, θ and φ; now, after the first rotation, ψ, when the axis X became X_1, any point, x_1, y_1, in the system thus obtained, would be expressed as follows in the X, Y, Z system:

$$X = x_1 \cos\psi - y_1 \sin\psi; \qquad Y = x_1 \sin\psi + y_1 \cos\psi;$$

where X_1 and Y_1 are the temporary coordinates of the point on the axes X_1 and Y_1 (this is evident from plain geometry). After the second rotation (through θ, about O–X_1), the axis Z took its final place z and the axis Y_1 moved up to, say, Y_2; under these conditions any point Y_2, z, X_1 will be expressed as follows in the system Y_1, Z, X_1:

$$Y_1 = Y_2 \cos\theta - z \sin\theta; \qquad z = Y_2 \sin\theta + z \cos\theta,$$

where X_1 remains the same as before, but Y_2 and z are the new coordinates in the system just obtained. Another, final, turn will now be made about z, through the angle φ, which finally gives any such point as x, y, z, in the system X_1, Y_2, z as follows,

$$X_1 = x \cos\varphi - y \sin\varphi; \qquad Y_2 = x \sin\varphi + y \cos\varphi$$

(z being the same as before). Eliminating X_1, Y_1, and Y_2 from these three pairs of equations we have

$$\begin{aligned} X &= x(\cos\psi \cos\varphi - \sin\psi \sin\varphi \cos\theta) \\ &\qquad - y(\cos\psi \sin\varphi + \sin\psi \cos\varphi \cos\theta) + z \sin\theta \sin\psi; \\ Y &= x(\cos\varphi \sin\psi + \sin\varphi \cos\psi \cos\theta) \\ &\qquad + y(\sin\psi \sin\varphi - \cos\psi \cos\varphi \cos\theta) - z \cos\psi \sin\theta; \\ Z &= x \sin\varphi \sin\theta + y \cos\varphi \sin\theta + z \cos\theta. \end{aligned}$$

These equations in themselves are not interesting, but the following deduction should be well retained by the reader for his future work: it is possible to find the coordinates of any point for the old system, solely through the Euler's angles and the new coordinates; but, *in finite form*, that is *involving no derivatives* (compare this with the result of our attempt to express a point in terms of the rotations, ω_1, ω_2 and ω_3 (see under Euler's formulae); it was shown then that x, y, z can also be given, in terms of ω_1, ω_2 and ω_3; only such expressions would necessarily involve velocities, time-derivatives).

Euler's angles are of great importance, therefore another example will be given: *A solid body, with one point fixed, is in*

motion which can be described as follows; the instantaneous axis is moving in space, and the projection of the constant angular velocity, Ω, about it (as laid off on that axis) upon a certain straight line, D, belonging to the body (and passing through the fixed point), is constant and $= \omega$; furthermore, the projection of the angular velocity, Ω, upon a plane perpendicular to D, is turning, relatively to the moving axes, x, y, z, with a constant angular velocity $= -\omega$. Required to determine, in terms of time, t, Euler's angles, with which the system can be located in relation to some fixed axes.

Let the line D (fig. 27) be the axis z of the moving system,

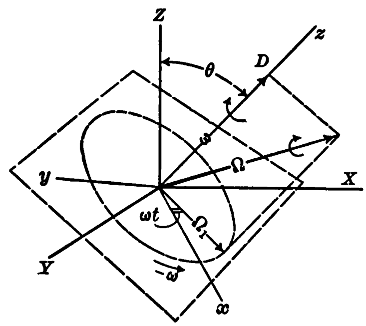


FIG. 27.

x, y, z (solid with the body); also let Ω_1 be the projection of Ω upon the plane perpendicular to D; let the moving axis x coincide with the projection of Ω at the beginning of the motion (that is when $t = 0$); finally let the fixed axes represent the position of moving axes at the beginning of motion, $t = 0$. From Euler's formulae just found we have

$$\omega_1 = \Omega_1 \cos \omega t = \frac{d\psi}{dt} \sin\theta \sin\varphi + \frac{d\theta}{dt} \cos\varphi,$$

$$\omega_2 = -\Omega_1 \sin \omega t = \frac{d\psi}{dt} \sin\theta \cos\varphi - \frac{d\theta}{dt} \sin\varphi, \tag{1}$$

$$\omega_3 = \Omega_1 = \frac{d\psi}{dt} \cos\theta + \frac{d\varphi}{dt}.$$

We have thus resolved the instantaneous velocity into its components along the moving axes, x, y, z; as a matter of fact the instantaneous velocity was given to us not by itself but by its projections; nevertheless, the formulae (1) will enable us to find $d\psi/dt$ (from the first two, multiplying by $\sin\varphi$ and by $\cos\varphi$ respectively, and adding)

$$\frac{d\psi}{dt} = \frac{\Omega_1 \sin(\varphi - \omega t)}{\sin\theta} \tag{2}$$

and also to find $d\theta/dt$ (multiplying the first by $\cos\varphi$, the second by $\sin\varphi$, and subtracting)

$$\frac{d\theta}{dt} = \Omega_1 \cos(\varphi - \omega t). \tag{3}$$

Also from the last equation (1),

$$\omega - \frac{d\varphi}{dt} = \frac{d\psi}{dt}\cos\theta = (\text{from } (2)) = \Omega_1 \sin(\varphi - \omega t)\frac{\cos\theta}{\sin\theta} \tag{4}$$

whence (multiplying (3) by (4))

$$\frac{\cos(\varphi - \omega t)}{\sin(\varphi - \omega t)}\left(\frac{d\varphi}{dt} - \omega\right) + \frac{\frac{d\theta}{dt}\cos\theta}{\sin\theta} = 0,$$

which immediately integrates into a logarithmic expression, from which

$$\sin(\varphi - \omega t)\sin\theta = \text{const.} = 0 \tag{5}$$

(the constant being $= 0$, because of our choice of coordinates, in view of which, when $t = 0$, $\theta = \theta_0 = 0$). The equation (5) can be satisfied only, when (1) $\theta = 0$ which means merely rotation about a fixed axis (z the same as Z); or, (2) when $\varphi = \omega t$; this means $d\varphi/dt = \omega$ and (see (4)) $d\psi/dt = 0$; whence $\psi = \text{const.} = 0$ (in view of our conditions). Finally from (3) we have $d\theta/dt = \Omega_1$ and $\theta = \Omega_1 t$ it appears, therefore, that the line D remains in the plane X-Z (since $\psi = 0$), while revolving about Y with constant angular velocity Ω_1

(since $\theta = \Omega_1 t$); and, meanwhile, the body is revolving about that axis with a constant velocity ω (since $\varphi = \omega t$).

We give one more example of the application of Euler's angles: *Consider the behavior of a gyroscope on the surface of the earth* (fig. 28).

Let Z be the vertical axis and X, Y, Z, the fixed axes; θ is one of Euler's angles, z—the axis of spin, and x and y are

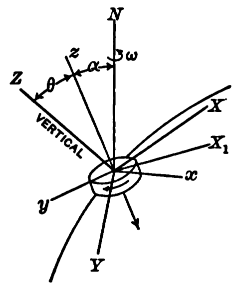


FIG. 28.

reference axes moving together with the body; let ω_{e_1}, ω_{e_2}, ω_{e_3} be the components along the same axes, x, y, z, of the rotation of the X, Y, Z system proper (owing to the rotation of the earth); the rotation of the gyroscope proper is ω_3; the rotation ω_e of the earth gives an additional component $\omega_{e_3} = -\omega_e \cos\alpha$. All acting forces (gravity and centrifugal) have resultants passing through the fixed point, so that their moments are O, and we can apply Euler's third equation

$$C\frac{d\omega_3}{dt} + (B - A)\omega_1\omega_2 = 0$$

C is here the moment of inertia of the gyroscope about its axis of spin; $A = B$ are two other moments of inertia, equal, and at right angles to the axis of spin; so that

$$C\frac{d}{dt}(\omega_3 - \omega_e \cos\alpha) = 0$$

or $\omega_3 - \omega_e \cos\alpha = \text{const.}$ If ω_{3_0} and α_0 are the initial values

of ω_3 and α, we have

$$\omega_3 = \omega_{3_0} + \omega_e(\cos\alpha - \cos\alpha_0)$$

in the absence of external work the kinetic energy (remembering that $2T = A\omega_1^2 + B\omega_2^2 + C\omega_3^2$, and that $A = B$) is constant:

$$A(\omega_1^2 + \omega_2^2) + C\omega_3^2 = A(\omega_{1_0}^2 + \omega_{2_0}^2) + C\omega_{3_0}^2$$

substituting ω_1 and ω_2 in terms of θ and ψ (see *transformation formulae*), as well as ω_3 just found, we have, neglecting ω_e^2 which is small,

$$A\left[\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{d\psi}{dt}\right)^2 \sin^2\theta\right]$$
$$= 2C\omega_{3_0}\omega_e(\cos\alpha_0 - \cos\alpha) + A(\omega_{1_0}^2 + \omega_{2_0}^2).$$

Now, in order to utilize the fact that there are no external moments about the direction N, fixed in space, we can state that the sum of the projections of the impulse axis on this direction is constant; these projections (instead of $A\omega_1$, $B\omega_2$, $C\omega_3$), are $A(\omega_1 + \omega_{e_1})\cos(N, x)$, etc., so that

$$A(\omega_1 + \omega_{e_1})\cos(N, x) + A(\omega_2 + \omega_{e_2})\cos(N, y)$$
$$+ C(\omega_3 + \omega_{e_3})\cos(N, z) = \text{const.}$$

But we can consider, at some given instant, the axis x as coinciding with the projection of N upon the equatorial plane of the gyroscope; in that case

$$\cos(N, x) = \sin\alpha; \qquad \cos(N, y) = 0; \qquad \cos(N, z) = \cos\alpha$$

also

$$\omega_{e_1} = -\omega_e\cos(Nx) = -\omega\sin\alpha; \qquad \omega_{e_2} = 0;$$
$$\omega_{e_3} = -\omega_e\cos\alpha$$

so that, finally

$$A(\omega_1 - \omega_e\sin\alpha)\sin\alpha + c(\omega_3 - \omega_e\cos\alpha)\cos\alpha = \text{const.};$$

or, substituting ω_3 from above

$$A(\omega_1 - \omega_e\sin\alpha)\sin\alpha + C(\omega_{3_0} - \omega_e\cos\alpha_0)\cos\alpha = \text{const.}$$

In order to find this constant let us re-write this equation for $t = 0$

$$A(\omega_1 \sin \alpha - \omega_{1_0} \sin \alpha_0) + A\omega_e(\sin^2 \alpha_0 - \sin \alpha_0)$$
$$+ C(\omega_{3_0} - \omega_e \cos \alpha_0)(\cos \alpha - \cos \alpha_0) = 0;$$

whence

$$A(\omega_1 \sin \alpha - \omega_{1_0} \sin \alpha_0) = (\cos \alpha_0 - \cos \alpha)[A\omega_e(\cos \alpha_0$$
$$+ \cos \alpha) + c(\omega_{3_0} - \omega_e \cos \alpha_0)].$$

This equation together with the integral of kinetic energy, found above, are the solutions of the problem. This is a rather abstract problem introduced here solely for the purpose of illustrating the application of Euler's angles. In what follows several easy problems will be given on gyroscopic motion.

(*Caution.*—The reader will understand that conceptions like *force, energy* or *work* are really never relative; for instance, the kinetic energy can be calculated merely as $\Sigma m(v_a{}^2/2)$ where v_a is the absolute velocity; although Koenig's theorem, which gives such an easy working rule, might be misinterpreted to mean that there is such a thing as relative kinetic energy in itself; we shall often refer to kinetic energy calculated relatively to a certain point or axis, as if they were fixed, but this should not mislead the reader; while of course relative velocity or relative path are perfectly real conceptions.)

CHAPTER II.

LAGRANGE'S EQUATIONS FOR A PARTICLE.

The position of a free material particle in space has so far been characterized by three coordinates, x, y, z, referred to three rectangular axes. When asked to determine the motion of a particle, we had to form the equations of motion and to derive from them the expressions of coordinates in terms of the time. Yet, mention has been made (see under *Coordinates of rigid body*) of a manner in which the number of coordinates can be reduced, owing to constraints; each constraining equation reducing the number of coordinates by unity; thus for example to determine the position of a particle in space three coordinates are needed, but, to determine the location of a point on the surface of the earth we need only two characteristics, longitude and latitude, the third characteristic being *implied* in the condition that the point is *on the sphere,* given by the equation of the latter. To make one step further, a point on the equator can be characterized by only one coordinate, such as an angle from some fixed point on the equator. What became of the other two characteristics? They have not been lost sight of or dropped altogether; they are implied in the doubly constraining condition; that the point is on the surface, and that it is in the equatorial plane, intersecting that surface. Hence it is easy to see that every additional constraining condition reduces the number of necessary characteristics or coordinates by one.

The characteristics which specify the position of a particle upon the given curve or surface are called *generalized* or *independent* coordinates. They are independent in the sense that in forming equations of motion involving these coordinates we can totally disregard the equations of constraints, unlike

in methods dealing with ordinary position coordinates. These independent coordinates can be of quite different nature from the usual position coordinates: for instance, in the example of the particle on the surface of a sphere, we have two independent coordinates, longitude and latitude, both being angular; in the example of a particle on the equator of a sphere we have only one such independent coordinate, the azimuthal angle from a given meridian; but we could likewise take as independent coordinate, say, a linear distance along the equator, from a given point. So that we can take as independent coordinates any characteristics, angular or linear, provided that they fully describe the position of the particle on its constraints. It is important to realize, however, that these independent or generalized coordinates, although less in number than ordinary position coordinates (one or two instead of three) lead to no ambiguity whatever in characterizing the position of a particle. In fact any one of the ordinary position coordinates, x, y, z, can at all times be expressed in terms of the new, independent, coordinates; and such expressions, if substituted in the equations of constraints, will render them identical. Take, for instance, the equation of the sphere $x^2 + y^2 + z^2 = r^2$; it is easy to see that the coordinates of any point, x, y, z, can be given in terms of the generalized coordinates, longitude and latitude, by the equations (fig. 29)

$$x = r \cos p \cos q,$$

$$y = r \cos p \sin q,$$

$$z = r \sin p,$$

where q is the longitude and p the latitude; substituting these equations into that of the sphere we have $r^2 = r^2$, an identity.

To recapitulate: generalized coordinates imply independence of the constraints; they are less in number than the position coordinates, x, y, z, yet the latter can always be expressed through them, and such expressions must necessarily satisfy

the constraints. The number of these generalized coordinates is called the *degree of freedom,* as we have already seen. In the case of a constraining curve, which is given as the intersection of two surfaces, we have only one degree of freedom; the

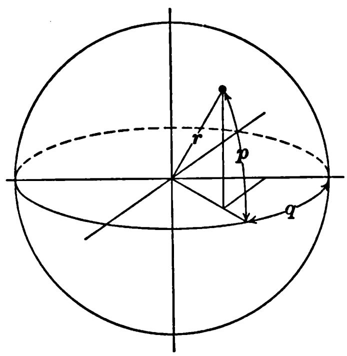


FIG. 29.

particle can move only along the curve and we need only one characteristic, that is, only one generalized coordinate, to completely locate its position on the curve; for instance in the case of a particle moving on the meridian of a sphere we had two equations $x^2 + y^2 + z^2 - r^2 = 0$ (the sphere); in general, say, $f_1(x, y, z) = 0$; and $z = 0$ (the equatorial plane); in general, say, $f_2(x, y, z) = 0$ so that only one generalized coordinate was necessary; we saw that it can be either an angle, reckoned from a given meridian, or a distance along the equator, from a certain initial point. And if we knew how to express this only variable, the generalized coordinate, which we shall call q, in terms of time, by some such equation as $q = f(t)$, we would know all about the motion of the particle, since the latter is fully located by this one variable, q. In other words, while in three-coordinate system we would have to know three such expressions, connecting coordinates, x, y, z, with time, t, $x = F_1(t)$; $y = F_2(t)$; $z = F_3(t)$, with our new method we need only one equation $q = f(t)$; afterward,

if desired, we can express the old position coordinates in terms of q, therefore obtaining the same equations, $x = F_1(t)$, etc.

Limiting ourselves, for the present, to the case of a particle moving along a curve given by two equations, $f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$ (which indicate *permanent constraints*, containing no time, t), let us see how the introduction of q, our new independent or generalized coordinate, will affect the shape of the fundamental equation of dynamics (from (1), under D'Alembert's principle), which, written for *one* particle of mass m, is

$$\left(X - m\frac{d^2x}{dt^2}\right)\delta x + \left(Y - m\frac{d^2y}{dt^2}\right)\delta y + \left(Z - m\frac{d^2z}{dt^2}\right)\delta z = 0$$

or

$$X\delta x + Y\delta y + Z\delta z = m\left(\frac{d^2x}{dt^2}\,\delta x + \frac{d^2y}{dt^2}\,\delta y + \frac{d^2z}{dt^2}\,\delta z\right), \quad (1)$$

the first part of which represents virtual work (see under *Virtual work*), and was denoted by δW. Knowing that x, y and z can ultimately be expressed in terms of our new variable q (see example above), by some such equations as

$$x = F_1(q); \qquad y = F_2(q); \qquad z = F_3(q); \qquad (2)$$

let us differentiate these equations with respect to time t; this will give velocities dx/dt, dy/dt, dz/dt in terms of q and of derivative dq/dt; in order to simplify writing we shall adopt the following notation; the time-derivatives will be indicated by (′) for each variable; so that dx/dt, dy/dt, dz/dt, dq/dt will be denoted simply as x', y', z', q'; and the values of these time derivatives will be (from (2))

$$x' = \frac{\partial F_1}{\partial q}\,q'; \qquad y' = \frac{\partial F_2}{\partial q}\,q'; \qquad z' = \frac{\partial F_3}{\partial q}\,q'; \qquad (3)$$

or simply

$$x' = \frac{\partial x}{\partial q} q'; \qquad y' = \frac{\partial y}{\partial q} q'; \qquad z' = \frac{\partial z}{\partial q} q'.$$

(The reader is advised to review his stock of knowledge of partial differentiation; see Bowser, Calculus, pp. 120, 122, 125.)

These values of x', y', z', if substituted in the general expression of kinetic energy

$$T = \frac{mv^2}{2} \quad \text{or} \quad 2T = m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]$$

will give

$$2T = m(x'^2 + y'^2 + z'^2) \tag{4}$$

which otherwise may be written thus (in view of (3))

$$2T = m\left[\left(\frac{\partial F_1}{\partial q}\right)^2 + \left(\frac{\partial F_2}{\partial q}\right)^2 + \left(\frac{\partial F_3}{\partial q}\right)^2\right] q'^2,$$

which is a quadratic function in q', whose coefficient, [ ], contains only q or its functions and constants; in other words

$$2T = \Phi(q) \times q'^2 \tag{4'}$$

which means that T is a function of q of constants and of the *square* of q' but contains no first power of q'.

Remembering that q itself is a function (as yet unknown) of the time, as is its derivative q', we can differentiate (4) with respect to q' (it will be partial differentiation, during which all variables other than q', or its functions, remain constant)

$$\frac{\partial T}{\partial q'} = m\left(x'\frac{\partial x'}{\partial q'} + y'\frac{\partial y'}{\partial q'} + z'\frac{\partial z'}{\partial q'}\right); \tag{5}$$

and also, with respect to q (partially),

$$\frac{\partial T}{\partial q} = m\left(x'\frac{\partial x'}{\partial q} + y'\frac{\partial y'}{\partial q} + z'\frac{\partial z'}{\partial q}\right). \tag{6}$$

To return to our fundamental equation (1): remembering how the equation (3) under *Virtual work* was derived, we can, by a similar application of Taylor's theorem, obtain the following (from (2)):

$$\delta x = \frac{\partial x}{\partial q}\delta q; \qquad \delta y = \frac{\partial y}{\partial q}\delta q; \qquad \delta z = \frac{\partial z}{\partial q}\delta q;$$

in view of which

$$X\delta x + Y\delta y + Z\delta z = \left(X\frac{\partial x}{\partial q} + Y\frac{\partial y}{\partial q} + Z\frac{\partial z}{\partial q}\right)\delta q;$$

or, putting

$$X\frac{\partial x}{\partial q} + Y\frac{\partial y}{\partial q} + Z\frac{\partial z}{\partial q} = Q, \tag{7}$$

we have simply

$$\delta W = X\delta x + Y\delta y + Z\delta z = Q\delta q.$$

Here δW is virtual work, while δq is the generalized displacement, and perfectly arbitrary, of our new independent coordinate; therefore Q cannot be other than some sort of a force, corresponding to such a displacement, since the product of the two represents work. Let us call Q *generalized force*. Its expression can be transformed as follows: In

$$X\frac{\partial x}{\partial q} + Y\frac{\partial y}{\partial q} + Z\frac{\partial z}{\partial q} = Q,$$

let us substitute, instead of the forces X, Y, Z, their equivalents, $m(d^2x/dt^2)$, $m(d^2y/dt^2)$, $m(d^2z/dt^2)$, and extend our simplified notation, so that

$$\frac{d^2x}{dt^2} = x''; \qquad \frac{d^2y}{dt^2} = y''; \qquad \frac{d^2z}{dt^2} = z'';$$

this will transform (7) into

$$m\left(x''\frac{\partial x}{\partial q} + y''\frac{\partial y}{\partial q} + z''\frac{\partial z}{\partial q}\right) = Q. \tag{8}$$

But this can be simplified still further; in the first place

$$x''\frac{\partial x}{\partial q}=\frac{d}{dt}\left(x'\frac{\partial x}{\partial q}\right)-x'\frac{d}{dt}\left(\frac{\partial x}{\partial q}\right), \tag{9}$$

which can be immediately verified by differentiation of $x'\cdot(\partial x/\partial q)$ with respect to time [the same remark applies to $y''(\partial y/\partial q)$ and $z''(\partial z/\partial q)$]; on the other hand, differentiating (3), partially, with respect to q' we have

$$\frac{\partial x'}{\partial q'}=\frac{\partial x}{\partial q} \tag{10}$$

with two similar equations in y and z. Also, differentiating (3) with respect to q we have

$$\frac{\partial x'}{\partial q}=\frac{\partial^2 x}{\partial q^2}q' \tag{11}$$

(because, since $q=f(t)$, q' is a function of t only), and two similar equations in y and z. Also, evidently

$$\frac{d}{dt}\left(\frac{\partial x}{\partial q}\right)=\frac{\partial^2 x}{\partial q^2}q', \text{ that is } =\frac{\partial x'}{\partial q} \tag{12}$$

(from (11)). Therefore, substituting (10) and (12) into (9) and, finally into (8) we have

$$\begin{aligned} m\left(x''\frac{\partial x}{\partial q}+y''\frac{\partial y}{\partial q}+z''\frac{\partial z}{\partial q}\right) \\ &= m\frac{d}{dt}\left(x'\frac{\partial x'}{\partial q'}+y'\frac{\partial y'}{\partial q'}+z'\frac{\partial z'}{\partial q'}\right) \\ &\quad - m\left(x'\frac{\partial x'}{\partial q}+y'\frac{\partial y'}{\partial q}+z'\frac{\partial z'}{\partial q}\right)=Q, \end{aligned} \tag{13}$$

which in view of (5) and (6) means

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'}\right)-\frac{\partial T}{\partial q}=Q. \tag{14}$$

Equations of this type are called *Lagrange's equations of*

motion; it will presently be shown that for the motion on a surface, where *two* independent coordinates are necessary to specify the position of a particle, there will be *two* such equations, one for each independent coordinate; also in the next chapter it will be explained how this same equation can be extended to a system.

The deduction of these equations is not difficult and not unnatural: all that we have done was to substitute new notations in the equation expressing that *kinetic energy* equals *work done;* this, in the first place, gave us a new expression of kinetic energy; on the other hand the "work done" has the same value, no matter what the choice of coordinates; it is always the product of force by displacement, and if, for some reason, we have chosen, instead of, say, δx, a peculiar displacement δq, we must expect that the corresponding (generalized) force will also be of a special nature, subject, however, to the condition that its product $Q\delta q$ is equal to the kinetic energy.

Now, we can imagine several kinds of displacements:

1. *Linear* displacement, the product of which by the corresponding *force* gives work.
2. *Angular* displacement, the product of which by the corresponding *moment* gives work.
3. Increase of *volume*, the product of which by the corresponding *pressure* gives work.

Etc.

So that what we called generalized force may be not only force, but moment, pressure, etc., according to the choice of the independent coordinate; it will be easily found from the conditions of the problem. If there are no external forces Q will be $= 0$; if there are external forces, the force Q must be found such, that it will correspond to the virtual displacement of our new (independent) coordinate (force for linear, moment for angular displacement, etc.). As a final result of these

transformations (and this was our primary object) *we have only one equation of motion* (14), *instead of three;* it can be written down as soon as the expression of kinetic energy T has been found; and has nothing to do with the constraints. Before illustrating the application of this equation by an example we can observe the following: in order to simplify the derivation of these equations we have made the assumption that the constraints $f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$ were permanent, that is that they contained no time, t. In general, however, the constraints may be variable in either shape or position, and therefore, as a general rule, their equations will be, say, $f_1(x, y, z, t) = 0$ and $f_2(x, y, z, t) = 0$, and consequently the equations (2) will also contain the time,

$$x = F_1(q, t); \qquad y = F_2(q, t); \qquad z = F_3(q, t).$$

By way of illustration imagine a particle on a sphere of varying diameter—a soap bubble. Here it is not possible to characterize the position of a particle by longitude and latitude alone, but time t must also be specified. We can easily prove, however, that moving constraints do not alter the form of Lagrange's equations; the kinetic energy will no longer be a function of q'^2 alone, but will have another term depending upon q'; but, although the equations (3) will have an additional term:

$$x' = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial q} q'; \ \cdots, \text{ etc.},$$

the equations (10) will still hold true, and in deriving the equations (12) we shall simply have

$$\frac{\partial x'}{\partial q} = \frac{\partial^2 x}{\partial t \partial q} + \frac{\partial^2 x}{\partial q^2} q',$$

but also $(d/dt)(\partial x/\partial q)$ will now be $(\partial^2 x/\partial t\partial q) + (\partial^2 x/\partial q^2)q'$ that is $= \partial x'/\partial q$, from what has just been obtained; so that the equation (14) will remain unchanged.

Having thus extended the method to the case of variable constraints we shall make another remark regarding the problem involving two parameters. The proof just given contemplates a particle with only one degree of freedom, that is to say, the motion is limited to a curve, given by two equations. Suppose now we have only one equation of constraints, $f(x, y, z) = 0$; this will represent a surface, so that two parameters or independent coordinates will be necessary to locate the position of the particle on that surface. In this case we can apply the same reasoning, except that two generalized coordinates will mean two generalized forces, and then there will be two Lagrange's equations, one for each independent coordinate. This can be applied to any number of coordinates, as we shall see in deriving Lagrange's equations for a system; but a particle can have only one, two or three degrees of freedom (in the latter case it is absolutely free of constraints). We shall rapidly review the derivation of equations similar to (14) for two independent (generalized) coordinates (such as latitude and longitude, for instance), which we shall call q_1 and q_2.

Let the constraining equation be $f(x, y, z) = 0$, and assume that the coordinates x, y, z, can be expressed in terms of the two independent coordinates thus:

$$x = F_1(q_1, q_2); \qquad y = F_2(q_1, q_2); \qquad z = F_3(q_1, q_2). \qquad (2')$$

Differentiating and substituting in the equation of kinetic energy we have

$$2T = Eq_1'^2 + Fq_1'q_2' + Gq_2'^2,$$

that is a quadratic function, homogeneous in the generalized velocities (q_1' and q_2'), where E, F, G, contain only generalized coordinates or constants.

Differentiating

$$2T = m(x'^2 + y'^2 + z'^2)$$

with respect to q_1 and q_1'; also to q_2 and q_2', we have two sets of equations like (5) and (6). But in this case the virtual displacement is

$$\delta x = \frac{\partial x}{\partial q_1}\delta q_1 + \frac{\partial x}{\partial q_2}\delta q_2,$$

(by Taylor's theorem) with similar expressions for δy and δz. Finally we have

$$\delta W = X\delta x + Y\delta y + Z\delta z = \left(X\frac{\partial x}{\partial q_1} + Y\frac{\partial y}{\partial q_1} + Z\frac{\partial z}{\partial q_1}\right)\delta q_1 + \left(X\frac{\partial x}{\partial q_2} + Y\frac{\partial y}{\partial q_2} + Z\frac{\partial z}{\partial q_2}\right)\delta q_2.$$

Here again we have generalized displacements δq_1 and δq_2; so that (δW being work) the coefficients of δq_1 and δq_2 must be generalized forces. Let

$$X\frac{\partial x}{\partial q_1} + Y\frac{\partial y}{\partial q_1} + Z\frac{\partial z}{\partial q_1} = Q_1,$$

and

$$X\frac{\partial x}{\partial q_2} + Y\frac{\partial y}{\partial q_2} + Z\frac{\partial z}{\partial q_2} = Q_2, \tag{7'}$$

so that $\delta W = Q_1\delta q_1 + Q_2\delta q_2$; repeating the method used above we obtain two equations like (8), one for q_1 and one for q_2.

Simplifying by the introduction of equations exactly similar to (9), (10) and (12) we finally have

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) - \frac{\partial T}{\partial q_1} &= Q_1, \\ \frac{d}{dt}\left(\frac{\partial T}{\partial q_2'}\right) - \frac{\partial T}{\partial q_2} &= Q_2. \end{aligned} \tag{15}$$

These are *Lagrange's equations of motion for a particle with two degrees of freedom.* For an entirely free particle we would have three such equations with three generalized forces, corresponding to the three generalized displacements; that is one equation for each independent coordinate. It will be

seen that they would be precisely alike in form and all derived from the same T; but the partial differentiation is performed with respect to the three different q's or q''s and the corresponding generalized forces, Q, will be different for each of these equations (15).

The case of varying constraints, for the reason given above, does not alter these equations; in that case instead of (2′) we shall have $x = f(q_1, q_2, t)$, etc.

Example. Form equations of motion of a free particle in a horizontal plane, under action of any force, P.

Here the only constraining equation is that of the plane, $z = 0$; the particle evidently has two degrees of freedom, and we shall choose as independent coordinates what otherwise are known as polar coordinates, r and φ; so that $q_1 = r$, and $q_2 = \varphi$. The kinetic energy in polar coordinates (see under *Integral of kinetic energy*) will be

$$2T = m\frac{dr^2 + r^2d\varphi^2}{dt^2}$$

or, in our simplified notation, $2T = m(r'^2 + r^2\varphi'^2)$.

Differentiating $2T$ with respect to r and r'; also to φ and φ', we have

$$\frac{\partial T}{\partial r} = mr\varphi'^2; \qquad \frac{\partial T}{\partial r'} = mr'; \qquad \frac{\partial T}{\partial \varphi} = 0; \qquad \frac{\partial T}{\partial \varphi'} = mr^2\varphi'.$$

Substituting in Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) - \frac{\partial T}{\partial q_1} = Q_1 \qquad \text{and} \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial q_2'}\right) - \frac{\partial T}{\partial q_2} = Q_2$$

we have

$$mr'' - mr\varphi'^2 = Q_1, \qquad \text{and} \qquad m\frac{d}{dt}(r^2\varphi') = Q_2.$$

These are the general equations of motion, except that Q_1 and Q_2 (generalized forces, corresponding to our generalized displacements δr and $\delta\varphi$) must be found from the original

conditions and substituted in these equations. Resolving the given force, P, into its (generalized) components, one, R, directed along the radius vector, the other, N, at right angles to the radius, we can observe the following: The force R is evidently $= Q_1$, because it is directed along the radial displacement δr, multiplied by which it gives $Q_1 \delta r$, that is work; the force N itself, if multiplied by the corresponding displacement, gives $N \delta \varphi$, which is not *work*, therefore N is not Q_2; we can see at once that Q_2 will not be a force but a moment, since the corresponding displacement is angular, and it is necessary to have a moment, in order to have *work*, when multiplied by an angular displacement; so that Q_2 is not N, but Nr.

The final equations of motion are

$$m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2\right] = R, \quad \text{and} \quad m\frac{d}{dt}\left(r^2\frac{d\varphi}{dt}\right) = N\cdot r. \qquad (16)$$

Example. Form general equations of motion of a free particle in space.

We take for independent coordinates r, θ and φ (see fig. 6 and corresponding remark on kinetic energy in polar coordinates). Here we have

$$2T = m[r'^2 + r^2(\theta'^2 + \cos^2\theta\cdot\varphi'^2)]$$

and there will be three Lagrange's equations for the three independent coordinates, r, θ and φ,

$$m[r'' - r(\theta'^2 + \cos^2\theta\cdot\varphi'^2)] = Q_1,$$

$$m\left[\frac{d}{dt}(r^2\theta') + r^2\sin\theta\cos\theta\cdot\varphi'^2\right] = Q_2,$$

$$m\frac{d}{dt}(r^2\cos^2\theta\cdot\varphi') = Q_3.$$

It remains only to substitute the corresponding values of the generalized forces Q_1, Q_2 and Q_3, in accordance with the

external forces as given in the problem. Suppose we have a force F which can be resolved into the following components:

R along r,
M perpendicular to r in the meridian plane,
P perpendicular to R and M,

then Q_1 will be the force R itself, because the displacement δr along it will be linear; but Q_2 will be Mr (because of the angular nature of the displacement q_2, or $\delta\theta$); also Q_3 will not be simply P but $Pr\cos\theta$, because the angular displacement $\delta\varphi$ will result in a certain work δW done by a force P acting on a lever $r\cos\theta$.

Finally

$$
\left.
\begin{aligned}
m\left\{\frac{d^2r}{dt^2} - r\left[\left(\frac{d\theta}{dt}\right)^2 + \cos^2\theta\left(\frac{d\varphi}{dt}\right)^2\right]\right\} &= R,\\
m\left[\frac{d}{dt}r^2\frac{d\theta}{dt} + r^2\sin\theta\cos\theta\left(\frac{d\varphi}{dt}\right)^2\right] &= Mr,\\
m\frac{d}{dt}r^2\cos^2\theta\frac{d\varphi}{dt} &= Pr\cos\theta.
\end{aligned}
\right\} \qquad (17)
$$

With a little practice the reader will be able to write down equations like (16) or (17) at once. Now the integration of equations thus obtained, or, in fact, of all but a very few simpler equations of dynamics, is very difficult, and often altogether impossible; but this difficulty is of a mathematical nature, while the formation of the equations is the real problem of dynamics. And in this connection Lagrange's equations are often of the greatest help.

An interesting case presents itself when the acting forces are the derivatives of some common force-function U. This symbol may be used in a broader sense than before, and may contain not only the coordinates but the time, t, explicitly (which the potential function we had before, could not), so that

$$
X = \frac{\partial U}{\partial x}; \qquad Y = \frac{\partial U}{\partial y}; \qquad Z = \frac{\partial U}{\partial z}. \qquad (18)
$$

If there is such a function, we can introduce (18) into (7), and obtain

$$Q = \frac{\partial U}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial q} + \frac{\partial U}{\partial z}\frac{\partial z}{\partial q} = \frac{\partial U}{\partial q}$$

so that instead of Q we will have, in equations (14), simply the partial derivatives, such as $\partial U/\partial q$, with respect to the corresponding independent coordinate. The same remark will apply to a greater number of parameters, so that, for instance, the equations (15) will then be re-written thus

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) - \frac{\partial T}{\partial q_1} = \frac{\partial U}{\partial q_1};$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_2'}\right) - \frac{\partial T}{\partial q_2} = \frac{\partial U}{\partial q_2}; \text{ and so on.}$$

The general rule of procedure when a force function can be found is as follows: first, we select the independent coordinates in such a manner that the x, y, z, coordinates can be expressed by the former in connection with the equations of constraints, which may also contain time; this will give equations (2) or (2′). The next step is to form T by the proper substitution of x', y', z', derived from (2) or (2′); the function T will thus contain time, t, as well as the independent coordinates and their time-derivatives; after this it remains to form the expression of generalized forces Q, or to substitute proper values of (2) or (2′) into the force function U, which will then contain only independent coordinates and, possibly, t. Thus we have everything required for writing down Lagrange's equations, one for each independent coordinate.

A few specially selected examples will now be given.

Example 1. *Find equations of motion of a particle, of mass m, moving without friction on a straight line, inclined at an angle α to a vertical axis* (fig. 30) *and rotating about the latter with the angular velocity ω.*

Comparing this with the equation of pendulum (Bowser, Anal. Mech., p. 458) we conclude that the motion of the particle will be pendular about A as a lowest point, and the time of an oscillation will be $= \pi/\omega$.

Example 3. *A particle starts with an initial velocity* v_0 (fig. 32) *along a circumference whose radius increases with the time; find the motion.*

This is a typical case of varying constraints (where $x = f(q, t)$, etc.). If the initial radius is a, the constraining equation will

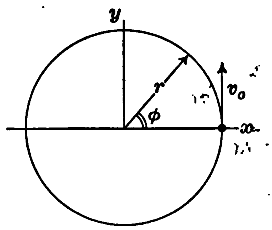


FIG. 32.

be $x^2 + y^2 = a^2(1 + kt)^2$, where k is a constant coefficient. Of course $z = 0$; also, introducing polar coordinates, r and φ, $x = a(1 + kt) \cos \varphi$ and $y = a(1 + kt) \sin \varphi$; differentiating with respect to both the independent coordinate and the time, t, we have

$$x' = -a(1 + kt)\varphi' \sin \varphi + ak \cos \varphi,$$

$$y' = a(1 + kt)\varphi' \cos \varphi + ak \sin \varphi,$$

which in $2T$ will give

$$2T = ma^2[(1 + kt)^2\varphi'^2 + k^2] = mv^2$$

(incidentally it follows that $a^2[(1 + kt)^2\varphi'^2 + k^2] = v^2$ at any time, or at the beginning of motion, when $t = 0$, $a^2\varphi'^2 + k^2 = v_0^2$ whence $a^2\varphi'^2 = v_0^2 - k^2$). From $2T$ we have

$$\frac{\partial T}{\partial \varphi'} = ma^2(1 + kt)^2\varphi',$$

so that

$$\frac{d}{dt}[ma^2(1+kt)^2\varphi'] = 0, \qquad \text{or} \qquad ma^2(1+kt)^2\varphi' = \text{const.} = c.$$

Let us determine the constant c for the beginning of motion, that is when $t = 0$; we have $ma^2\varphi' = c$; but $a^2\varphi'^2 = v_0^2 - a^2k^2$, so that $a\varphi' = \sqrt{v_0^2 - a^2k^2}$; whence $ma^2\varphi' = ma\sqrt{v_0^2 - a^2k^2} = c$. Finally our equation becomes

$$a(1+kt)^2\varphi' = \sqrt{v_0^2 - a^2k^2} \qquad \text{or} \qquad \frac{d\varphi}{dt} = A\frac{1}{(1+kt)^2},$$

where

$$A = \frac{\sqrt{v_0^2 - a^2k^2}}{a}.$$

Separating the variables we have,

$$d\varphi = A\frac{dt}{(1+kt)^2},$$

and finally

$$\varphi = -\frac{A}{k}\frac{1}{1+kt} + C_1.$$

But for $t = 0$, $\varphi = 0$, and $C_1 = (A/k)$, so that

$$\varphi = \frac{A}{k}\left(1 - \frac{1}{1+kt}\right) = \frac{A}{k}\frac{kt}{1+kt} = \frac{At}{1+kt}.$$

Substituting this into x and y, formed in the beginning of the problem, we have

$$x = a(1+kt)\cos\frac{At}{1+kt},$$

$$y = a(1+kt)\sin\frac{At}{1+kt},$$

these are the final equations of motion (and completely express the coordinates in terms of the time).

Remark 1. Due mention has been made of the ease with which generalized forces can be found if there is a force function U; the latter may be a potential function, that is a function of the coordinates only; or, it may contain the time; but, in any case, its partial derivatives, with respect to any coordinate, must give corresponding forces, so that

$$\frac{\partial U}{\partial x} = X, \text{ etc.};$$

instead of generalized forces Q, etc., we then simply have $\partial U/\partial q$, etc.

But if there is no force function, the generalized forces Q can be found from general considerations as follows: We have seen (equation (7)) that

$$Q = X\frac{\partial x}{\partial q} + Y\frac{\partial y}{\partial q} + Z\frac{\partial z}{\partial q};$$

so that it is only necessary to calculate the partial derivatives $\partial x/\partial q$, etc. from the equations (2) or (2′), express x, y, z through the independent coordinates, and substitute in (7); the result will be Q for that specific displacement. Partial differentiation means assuming other generalized displacements, all $= 0$, and the virtual work, $Q\delta q$, deriving *only* from the displacement we have under present consideration, say δq; similarly for Q_2 we shall find $\partial x/\partial q_2$, etc.; and so on; X, Y, and Z, being of course the projections of external forces upon the axes, x, y, z.

Remark 2. Should it be desired to calculate the reactions of the constraints, this can be easily done by referring to equation (4) under *D'Alembert's principle,* where λ_1, λ_2, etc., are Lagrange's multipliers, or as they are often called, constraint-coefficients. The working of this method will be illustrated presently.

Remark 3. We have seen, under *Integral of kinetic energy,* that, for constrained motion, if the constraints are permanent or independent of the time, the integral of kinetic energy

$T = U + h$ holds as true as for non-constrained motion; this integral can also be shown to be an immediate consequence, of and therefore an equivalent to, of any of Lagrange's equations. Considering the simplest case of motion, with one degree of freedom, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'}\right) - \frac{\partial T}{\partial q} = Q;$$

multiplying by q' we have

$$q'\frac{d}{dt}\left(\frac{\partial T}{\partial q'}\right) - q'\frac{\partial T}{\partial q} = Qq'. \tag{19}$$

But $q'(d/dt)(\partial T/\partial q')$ is the same as $(d/dt)(q'(\partial T/\partial q')) - q''(\partial T/\partial q')$, as can immediately be verified by differentiating $q'(\partial T/\partial q')$ with respect to the time; therefore

$$Qq' = \frac{d}{dt}\left(q'\frac{\partial T}{\partial q'}\right) - q''\frac{\partial T}{\partial q'} - q'\frac{\partial T}{\partial q}. \tag{20}$$

Remembering the remark made under (4′) we can differentiate $2T = \phi(q)q'^2$, with respect to q', and the result (which holds true for permanent constraints only) will be

$$2\frac{\partial T}{\partial q'} = 2\phi(q)\cdot q',$$

whence

$$q'\frac{\partial T}{\partial q'} = \phi(q)\cdot q'^2 = 2T. \tag{21}$$

From the same expression, for $2T$, not containing time explicitly, we have

$$\frac{dT}{dt} = \frac{\partial T}{\partial q}q' + \frac{\partial T}{\partial q'}q'';$$

which substituted in (20) and then in (19) will give

$$\frac{d}{dt}(2T) - \frac{dT}{dt} = \frac{dT}{dt} = Qq' = Q\frac{dq}{dt}$$

whence $dT = Qdq$, which is the same as (2) under *Integral of kinetic energy.*

In the case of several degrees of freedom, we have, in the same manner,

$$\frac{dT}{dt} = Q_1q_1' + Q_2q_2' + Q_3q_3',$$

which likewise leads to the integral of kinetic energy. If there is a force function U of q_1, q_2, q_3, of which $Q_1q_1 + Q_2q_2 + Q_3q_3$ is an exact differential, dU, then we simply have $dT = dU$, so that $T = U + h$; this is the case when there is a potential function, $U(x, y, z)$, not containing the time. Hence in problems involving the potential function it is perfectly legitimate to replace *any* Lagrange's equation by its equivalent, the *integral of kinetic energy.*

The most difficult equations can thus be done away with and the remaining set, including the integral of kinetic energy, will then be the final equations of motion.

Example. By way of illustration of these remarks let us take the case of *a particle moving, under gravity, on a sphere of constant radius r.*

We had a similar problem (see equations (17)) for a more general case of motion; as before, let us take as independent coordinates the latitude θ and the longitude φ. Through these, our original coordinates can be expressed as follows:

$$x = r \cos\theta \cos\varphi; \qquad y = r \cos\theta \sin\varphi; \qquad z = r \sin\theta;$$

and, r being constant, the kinetic energy will be

$$2T = m(r^2\theta'^2 + r^2 \cos^2\theta.\varphi'^2).$$

Writing Lagrange's equations, one for θ and one for φ, we have

$$mr(\theta'' + \sin\theta \cos\theta\varphi'^2) = \Theta, \qquad \text{and} \qquad m\frac{d}{dt}(\cos^2\theta\varphi') = \Phi;$$

where Θ and Φ are generalized forces which we shall now proceed to determine (see remark 1). The axis z being directed upward, the potential function will be

$$U = -mgz = -mgr\sin\theta;$$

forming $\partial U/\partial\theta$ we have

$$\frac{\partial U}{\partial\theta} = \Theta = -mgr\cos\theta;$$

also

$$\frac{\partial U}{\partial\varphi} = \Phi = 0;$$

so that the final equations will be

$$\frac{d^2\theta}{dt^2} + \sin\theta\cos\theta\varphi'^2 = -g\cos\theta, \quad \text{and} \quad \frac{dt}{dt}\cos^2\theta\frac{d\varphi}{dt} = 0. \quad (22)$$

From the latter we have

$$\cos^2\theta\frac{d\varphi}{dt} = c,$$

which is the integral of areas for $\theta = 0$ (that is for the plane x–y).

If instead of a potential function U we had a force whose projections upon the axes were X, Y, Z, we would find Θ and Φ as follows: In view of the remark 1 we would have

$$\Theta = X\frac{\partial x}{\partial\theta} + Y\frac{\partial y}{\partial\theta} + Z\frac{\partial z}{\partial\theta},$$

but

$$x = r\cos\theta\cos\varphi; \qquad y = r\cos\theta\sin\varphi; \qquad z = r\sin\theta;$$

so that

$$\frac{\partial x}{\partial\theta} = -r\sin\theta\cos\varphi; \qquad \frac{\partial y}{\partial\theta} = -r\sin\theta\sin\varphi; \qquad \frac{\partial z}{\partial\theta} = r\cos\theta.$$

Substituting in the expression of Θ we would find it in terms

of the (known) projections X, Y, Z, and of the parameters θ and φ; Φ would be found in the like manner. (Instead of this we might at once have written down Θ and Φ in terms of the components of F, tangent to the sphere in the meridian plane, M; and along the tangent to the parallel circle, P (see equations (17)). But our object was to indicate a general method for obtaining the Q's.)

According to Remark 3, we can replace the first equation (22) by the integral of kinetic energy, which is $T = U + h$. In our problem $U = - mgr \sin \theta$ and

$$T = \frac{m}{2} (r^2\theta'^2 + r^2 \cos^2 \theta \cdot \varphi'^2)$$

so that the final equation $T - U = h$ will be

$$r^2\theta'^2 + r^2 \cos^2 \theta \cdot \varphi'^2 + 2mgr \sin \theta = 2h$$

(where h is a constant); this equation, together with the second equation (22), will determine the motion; it is only necessary to substitute φ' from the latter into the former and to integrate the result. Such integration is not easy, although it can actually be performed.

The reaction of the constraints (in our case, of the spherical surface on which the particle is constrained to move, or else of the rod (length $= r$), fixed in the center of the sphere, which can perform precisely the same function as would the sphere), according to Remark 2 can be found as follows: The result of integrating the equations of motion (22) will give us θ and φ in terms of time, t; as soon as we have θ and φ, we can substitute them in the expressions of x, y and z, which will give x, y and z in terms of t; then the constraints' coefficient λ can be calculated from the general equation

$$m \frac{d^2x}{dt^2} = X + \lambda \frac{\partial f}{\partial x}$$

(with similar equations for y and z), where X is the projection

of the acting force and f is the constraining equation (in our case the equation of the sphere, $x^2 + y^2 + z^2 - r^2 = 0$). Since gravity is the only acting force, we have $X = 0$; $Y = 0$, and the only projection is $Z = -mg$; so that λ can be derived from any of these three equations

$$m\frac{d^2x}{dt^2} + \lambda x = 0; \qquad m\frac{d^2y}{dt^2} + \lambda y = 0; \qquad m\frac{d^2z}{dt^2} - g + \lambda z = 0.$$

If the motion were on a curve instead of a surface, we would have two constraining surfaces; and two coefficients λ_1 and λ_2 could be derived from any two equations of motion; but in our specific problem a much easier method can be used. Mention has already been made (see the problem of a particle sliding off a cylinder, under *Integral of kinetic energy*) of the fact that the pressure exerted by a particle describing a curve is equal to the centrifugal force minus the radial component due to the applied force; in our problem the centrifugal force is mv^2/r, and v^2 can be derived from the equation of energy, $T = U + h$, or $v^2 = 2(h - gz)$; on the other hand it is easy to see that the component of the weight, mg, upon the radius will be $= mgz/r$ so that the reaction

$$P = \frac{m}{r}(2h - 3gz),$$

h being a constant.

Remark on relative motion. We need not dwell on the subject of relative motion in studying the application of Lagrange's equations to a single particle, for the reason that here the kinetic energy, which is the first item to be calculated, can easily be found. Indeed, we have seen that Coriolis's acceleration, which is the most characteristic attribute of relative motion, does not perform any work, being at right angles to the relative path; hence it does not appear in the expression of kinetic energy, which is based upon absolute velocity, or, according to Koenig's theorem, can be derived

as a sum of the kinetic energy that *would* be due to (*mar*) motion alone, and of the kinetic energy that *would* be due to relative motion alone. In studying the motion of a system (two or more particles, rigid body, etc.) which will be done in the next chapter, we shall have several new principles to establish in connection with relative motion; but the word *system* implies such things as principal axes, moments of inertia, etc., all of which will complicate our problem quite considerably and, incidentally, make it most fascinating. But in dealing with a particle we have no such complications; for instance, the problem of a particle, moving freely in a circular tube, revolving in a horizontal plane, which is really a typical problem on relative motion, was solved by us without any reference to relative motion: having selected φ as our independent variable, we wrote down the expressions of absolute coordinates, x and y, in terms of φ. After this, an easy differentiation and substitution in the expression of $2T$ gave us the desired expression of kinetic energy, and the solution, which showed that the motion of the particle would be pendular, gave us merely the dependence of φ upon the time, which made it purely relative, that is such as it would appear to an

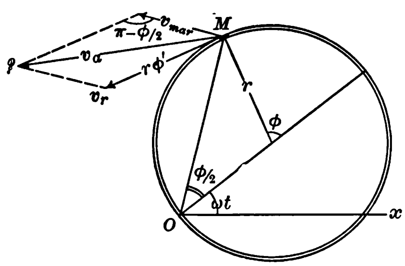


FIG. 33.

observer, connected to (and traveling with) the moving system, or ring. But of course it is just as easy to treat the problem from the standpoint of relative motion: we can find the absolute velocity v_a as the geometric sum of two velocities,

v_{mar} and v_r; after which v_a can be substituted in $mv_a^2/2$; thus, v_r is evidently the velocity along the circumference, $= r \times$ ang. vel. $= r\varphi'$; while the v_{mar} is the velocity of the point M (see fig. 33) rotating about O with the angular velocity ω of the reference system (of the circumference itself). Adding these we obtain v_a^2 and therefore the kinetic energy. We shall ask the reader to check this in accordance with the drawing and to show that the kinetic energy so obtained is precisely the same as before.

CHAPTER III.

LAGRANGE'S EQUATIONS FOR A SYSTEM.

By a system we mean two or more particles, connected in some manner; or, more particularly, a rigid body, a great multitude of particles subject to the one essential condition, permanence of form; or, a combination of such bodies. In deriving Lagrange's equations for a system we shall be governed by the great principle of virtual work, in view of which, if we know, for any virtual displacement, the total amount of work done by all forces, we have all that is necessary for finding the motion. Indeed, the conception of work enables us to form the expression of kinetic energy of the system, and from the latter, as we have seen before, and shall again re-establish presently, in a different manner, the equations of motion can be derived. Of course it is almost evident that what has been said regarding a particle can be extended to a system, but we shall derive Lagrange's equations without reference to what has been said in the last chapter. However, in following the mechanism of derivation of these equations, it is well to constantly bear in mind Lagrange's fundamental idea: the general equation of dynamics can be split into two parts, one of which, representing virtual work, consists of a coefficient (say Q) multiplied by the virtual displacement. Of course for the same virtual displacement, δq, the coefficient Q must be the same, regardless of what particular system of coordinates x, y, z, we have chosen. Therefore, the other side of the equation will likewise be independent of the coordinates in the same sense. But, kinetic energy can be expressed in terms of $q, \cdots$, etc., $q', \cdots$, etc. (and of t, sometimes) and is independent of the choice of coordinates x, y, z. Hence, the natural conclusion that the first member of the equation just referred to must be made derivable from T.

Lagrange's beautiful transformation shows how this can be done.

Consider a system consisting of n separate particles whose coordinates are given in rectangular coordinates by $x_1, y_1, z_1, x_2, y_2, z_2, \cdots, x_n, y_n, z_n$; each particle is actuated by forces, of which let the components upon the axes be $X_1, Y_1, Z_1, X_2, Y_2, Z_2, \cdots, X_n, Y_n, Z_n$.

Then the fundamental equation of dynamics, for a system comprising all such particles, will be

$$\Sigma\left[\left(X - m\frac{d^2x}{dt^2}\right)\delta x + \left(Y - m\frac{d^2y}{dt^2}\right)\delta y + \left(Z - m\frac{d^2z}{dt^2}\right)\delta z\right] = 0;$$

which can be re-arranged thus:

$$\Sigma(X\delta x + Y\delta y + Z\delta z) = \Sigma m\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z\right). \quad (1)$$

Adding to this equation the equations of constraints we have all that is necessary for finding the motion of the system. Let us select k independent variables $q_1, q_2, \cdots, q_k$ (this number k will be equal to $3n - m$, where n is the number of particles, and m of constraining equations, as we have seen before) by which any coordinates $x_1, y_1, z_1, x_2, y_2, z_2, \cdots$, etc., can be expressed in some such manner as

$$\begin{aligned}
x_1 &= f_1(q_1, q_2, \cdots, q_k, t),\\
y_1 &= f_2(q_1, q_2, \cdots, q_k, t),\\
z_1 &= f_3(q_1, q_2, \cdots, q_k, t),\\
x_2 &= f_4(q_1, q_2, \cdots, q_k, t),\\
y_2 &= f_5(q_1, q_2, \cdots, q_k, t),\\
z_2 &= f_6(q_1, q_2, \cdots, q_k, t),\\
&\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\\
x_n &= f_{3n-2}(q_1, q_2, \cdots, q_k, t),\\
y_n &= f_{3n-1}(q_1, q_2, \cdots, q_k, t),\\
z_n &= f_{3n}(q_1, q_2, \cdots, q_k, t).
\end{aligned} \quad (2)$$

In other words every coordinate (three for each particle) can be expressed, in terms of the independent variables, and this may and, generally does, also involve time, t (that is, including the most general case, that of movable constraints). If the independent coordinates have been chosen in accordance with the equations of constraints, the latter will be satisfied by a substitution of (2) therein. Our object is to transform (1) to new variables q_1, q_2, $\cdots$, etc. The first part of (1) can be easily transformed thus

$$\Sigma(X\delta x + Y\delta y + Z\delta z) = Q_1\delta q_1 + Q_2\delta q_2 + \cdots Q_k\delta q_k. \tag{3}$$

Indeed, from (2) we have (by developing $x + \delta x$ by Taylor's theorem in the same manner as has been done before),

$$\begin{aligned}
\delta x_1 &= \frac{\partial x_1}{\partial q_1}\delta q_1 + \frac{\partial x_1}{\partial q_2}\delta q_2 + \cdots + \frac{\partial x_1}{\partial q_k}\delta q_k,\\
\delta y_1 &= \frac{\partial y_1}{\partial q_1}\delta q_1 + \frac{\partial y_1}{\partial q_2}\delta q_2 + \cdots + \frac{\partial y_1}{\partial q_k}\delta q_k,\\
\delta z_1 &= \frac{\partial z_1}{\partial q_1}\delta q_1 + \frac{\partial z_1}{\partial q_2}\delta q_2 + \cdots + \frac{\partial z_1}{\partial q_k}\delta q_k,\\
\delta x_2 &= \frac{\partial x_2}{\partial q_1}\delta q_1 + \frac{\partial x_2}{\partial q_2}\delta q_2 + \cdots + \frac{\partial x_2}{\partial q_k}\delta q_k,\\
\delta y_2 &= \frac{\partial y_2}{\partial q_1}\delta q_1 + \frac{\partial y_2}{\partial q_2}\delta q_2 + \cdots + \frac{\partial y_2}{\partial q_k}\delta q_k,\\
\delta z_2 &= \frac{\partial z_2}{\partial q_1}\delta q_1 + \frac{\partial z_2}{\partial q_2}\delta q_2 + \cdots + \frac{\partial z_2}{\partial q_k}\delta q_k, \qquad (4)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\delta x_n &= \frac{\partial x_n}{\partial q_1}\delta q_1 + \frac{\partial x_n}{\partial q_2}\delta q_2 + \cdots + \frac{\partial x_n}{\partial q_k}\delta q_k,\\
\delta y_n &= \frac{\partial y_n}{\partial q_1}\delta q_1 + \frac{\partial y_n}{\partial q_2}\delta q_2 + \cdots + \frac{\partial y_n}{\partial q_k}\delta q_k,\\
\delta z_n &= \frac{\partial z_n}{\partial q_1}\delta q_1 + \frac{\partial z_n}{\partial q_2}\delta q_2 + \cdots + \frac{\partial z_n}{\partial q_k}\delta q_k;
\end{aligned}$$

substituting these in (3) and factoring by $\delta q_1, \delta q_2, \cdots, \delta q_k$ we have

$$
\begin{aligned}
\Sigma(X\delta x + Y\delta y + Z\delta z)
&= \left(X_1\frac{\partial x_1}{\partial q_1} + Y_1\frac{\partial y_1}{\partial q_1} + Z_1\frac{\partial z_1}{\partial q_1} + X_2\frac{\partial x_2}{\partial q_1} + Y_2\frac{\partial y_2}{\partial q_1}\right.\\
&\qquad \left. + \cdots + X_n\frac{\partial x_n}{\partial q_1} + Y_n\frac{\partial y_n}{\partial q_1} + Z_n\frac{\partial z_n}{\partial q_1}\right)\delta q_1\\
&+ \left(X_1\frac{\partial x_1}{\partial q_2} + Y_1\frac{\partial y_1}{\partial q_2} + Z_1\frac{\partial z_1}{\partial q_2}\right.\\
&\qquad \left. + \cdots + X_n\frac{\partial x_n}{\partial q_2} + Y_n\frac{\partial y_n}{\partial q_2} + Z_n\frac{\partial z_n}{\partial q_2}\right)\delta q_2\\
&+ \cdots\cdots\cdots\cdots\cdots\cdots\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\\
&+ \left(X_1\frac{\partial x_1}{\partial q_k} + Y_1\frac{\partial y_1}{\partial q_k} + Z_1\frac{\partial z_1}{\partial q_k}\right.\\
&\qquad \left. + \cdots + X_n\frac{\partial x_n}{\partial q_k} + Y_n\frac{\partial y_n}{\partial q_k} + Z_n\frac{\partial z_n}{\partial q_k}\right)\delta q_k;
\end{aligned}
$$

or, denoting the coefficients of $\delta q_1, \delta q_2, \cdots, \delta q_k$, by $Q_1, Q_2, \cdots, Q_k$, we have simply

$$\Sigma(X\delta x + Y\delta y + Z\delta z) = Q_1\delta q_1 + Q_2\delta q_2 + \cdots + Q_k\delta q_k,$$

that is (3), where

$$
\begin{aligned}
Q_1 &= X_1\frac{\partial x_1}{\partial q_1} + Y_1\frac{\partial y_1}{\partial q_1} + Z_1\frac{\partial z_1}{\partial q_1} + X_2\frac{\partial x_2}{\partial q_1} + \cdots + Z_n\frac{\partial z_n}{\partial q_1};\\
Q_2 &= X_1\frac{\partial x_1}{\partial q_2} + Y_1\frac{\partial x_1}{\partial q_2} + Z_1\frac{\partial z_1}{\partial q_2} + \cdots + Z_n\frac{\partial z_n}{\partial q_2}; \qquad (5)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
Q_k &= X_1\frac{\partial x_1}{\partial q_k} + Y_1\frac{\partial y_1}{\partial q_k} + Z_1\frac{\partial z_1}{\partial q_k} + \cdots + Z_n\frac{\partial z_n}{\partial q_k}.
\end{aligned}
$$

We know that (3) represents virtual work and we had a special notation for it, δW. (It is well to observe, however,

that, in general, there is no such function as W, the exact variation of which is (3), so that δW is merely a symbol, introduced in order to simplify writing.)

From (3) it follows that all terms like $Q \cdot \delta q$ are elements of work; and, whatever the nature of q, if δq is a generalized displacement, its coefficient, Q, must be a corresponding force, which we shall agree to call generalized force. (As we have seen this may be not only force but moment, pressure, etc., according to the nature of the independent variable q itself.)

It only remains to transform the second member of (1), so as to make it involve only the kinetic energy T. In the first place, using the notation $d^2x/dt^2 = x''$, etc., and considering (4) we can rewrite the second member thus (where the summation includes all q's, from q_1 to q_k inclusive)

$$\Sigma m\left(x''\frac{\partial x}{\partial q} + y''\frac{\partial y}{\partial q} + z''\frac{\partial z}{\partial q}\right)\delta q = \delta W. \tag{6}$$

On the other hand the expression of kinetic energy is

$$2T = \Sigma m(x'^2 + y'^2 + z'^2) \tag{7}$$

but, from (2) we have (remembering that the variables are $q_1, q_2, \cdots, q_k$, and also the time t)

$$\begin{aligned}
x_1' &= \frac{\partial x_1}{\partial t} + \frac{\partial x_1}{\partial q_1}q_1' + \frac{\partial x_1}{\partial q_2}q_2' + \cdots + \frac{\partial x_1}{\partial q_k}q_k', \\
y_1' &= \frac{\partial y_1}{\partial t} + \frac{\partial y_1}{\partial q_1}q_1' + \frac{\partial y_1}{\partial q_2}q_2' + \cdots + \frac{\partial y_1}{\partial q_k}q_k', \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
z_{3n} &= \frac{\partial z_{3n}}{\partial t} + \frac{\partial z_{3n}}{\partial q_1}q_1' + \frac{\partial z_{3n}}{\partial q_2}q_2' + \cdots + \frac{\partial z_{3n}}{\partial q_k}q_k',
\end{aligned} \tag{8}$$

from which it also follows (by partial differentiation as to $q_1, q_2, \cdots, q_k$) that

$$\frac{\partial x_1'}{\partial q_1'} = \frac{\partial x_1}{\partial q_1}; \qquad \frac{\partial y_1'}{\partial q_1'} = \frac{\partial y_1}{\partial q_1}; \qquad \cdots \qquad \frac{\partial z_{3n}'}{\partial q_k'} = \frac{\partial z_{3n}}{\partial q_k}. \tag{9}$$

Differentiating (7) with respect to any q we have

$$\frac{\partial T}{\partial q} = \Sigma m\left(x'\frac{\partial x'}{\partial q} + y'\frac{\partial y'}{\partial q} + z'\frac{\partial z'}{\partial q}\right), \tag{10}$$

while the derivative of (7) with respect to any q' will be (in view of (9))

$$\frac{\partial T}{\partial q'} = \Sigma m\left(x'\frac{\partial x}{\partial q} + y'\frac{\partial y}{\partial q} + z'\frac{\partial z}{\partial q}\right). \tag{11}$$

This can be differentiated with respect to t [remembering that

$$\frac{d}{dt}\left(\frac{\partial x}{\partial q}\right) = \frac{\partial x'}{\partial q}$$

for any x; which can be easily seen by differentiating say (8) with respect to *any* q

$$\frac{\partial x_1{}'}{\partial q} = \frac{\partial^2 x_1}{\partial t \partial q} + \frac{\partial^2 x_1}{\partial q^2} q' + \cdots.$$

Also differentiating $\partial x_1/\partial q$ (from (2)) with respect to t:

$$\frac{\partial}{\partial t}\frac{\partial x_1}{\partial q} = \frac{\partial^2 x_1}{\partial t \partial q} + \frac{\partial^2 x_1}{\partial q^2} q' + \cdots$$

and comparing both results]; so that (for any q)

$$\frac{d}{dt}\frac{\partial T}{\partial q'} = \Sigma m\left(x'\frac{\partial x'}{\partial q} + y'\frac{\partial y'}{\partial q} + z'\frac{\partial z'}{\partial q} + x''\frac{\partial x}{\partial q} + y''\frac{\partial y}{\partial q} + z''\frac{\partial z}{\partial q}\right)$$

or, subtracting (10) and in view of (6) and (3)

$$\Sigma\left(\frac{d}{dt}\frac{\partial T}{\partial q'} - \frac{\partial T}{\partial q}\right)\delta q = \delta W = \Sigma Q \delta q \cdots. \tag{12}$$

Assuming that all q_1, q_2, $\cdots$, are independent, the only manner in which (12) can be satisfied is by equating the

coefficients of each δq. Whence

$$\begin{aligned} \frac{d}{dt}\frac{\partial T}{\partial q_1'} - \frac{\partial T}{\partial q_1} &= Q_1, \\ \frac{d}{dt}\frac{\partial T}{\partial q_2'} - \frac{\partial T}{\partial q_2} &= Q_2, \\ \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \quad \cdot \\ \frac{d}{dt}\frac{\partial T}{\partial q_k'} - \frac{\partial T}{\partial q_k} &= Q_k. \end{aligned} \tag{13}$$

There will be k such equations, one for each independent coordinate. The generalized forces Q_1, Q_2, $\cdots$, etc., are given by the k equations (5). Any desired generalized force, say Q_m can be calculated from the general expression (see (5))

$$\begin{aligned} Q_m = X_1\frac{\partial x_1}{\partial q_m} + Y_1\frac{\partial y_1}{\partial q_m} + Z_1\frac{\partial z_1}{\partial q_m} + \cdots \\ + X_n\frac{\partial x_n}{\partial q_m} + Y_n\frac{\partial x_n}{\partial q_m} + Z_n\frac{\partial z_n}{\partial q_m}; \end{aligned} \tag{14}$$

by assuming that all other displacements, δq, corresponding to other forces Q are temporarily made equal to O, so that all the work done by the forces X_1, Y_1, Z_1, $\cdots$, X_n, Y_n, Z_n, is represented exclusively by $Q_m \delta q_m$.

All we have to do is to find all the necessary derivatives $(\partial x_1/\partial q_m)$, $\cdots$, $(\partial z_n/\partial q_m)$, from (2) and to substitute them in (14). We shall return to this in solving examples. However, it often happens that the given forces derive from some force function U; this may be the potential function as we had before, or, again, it may contain time (which the potential function does not). At any rate such a function possesses the property of having its partial derivatives, with respect to any coordinate, equal to the corresponding forces; so that for instance

$$X_1 = \frac{\partial U}{\partial x_1}; \qquad Y_1 = \frac{\partial U}{\partial y_1}; \qquad \cdots \qquad Z_n = \frac{\partial U}{\partial z_n};$$

in that case instead of (5) or, say, (14) we have

$$Q_m = \frac{\partial U}{\partial x_1}\frac{\partial x_1}{\partial q_m} + \frac{\partial U}{\partial y_1}\frac{\partial y_1}{\partial q_m} + \cdots + \frac{\partial U}{\partial z_n}\frac{\partial z_n}{\partial q_m},$$

which is simply $= \partial U/\partial q_m$; so that, instead of the equation (13) we shall have, in this special case,

$$\left.\begin{array}{c} \dfrac{d}{dt}\dfrac{\partial T}{\partial q_1'} - \dfrac{\partial T}{\partial q_1} = \dfrac{\partial U}{\partial q_1}, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \dfrac{d}{dt}\dfrac{\partial T}{\partial q_k'} - \dfrac{\partial T}{\partial q_k} = \dfrac{\partial U}{\partial q_k}. \end{array}\right. \qquad (15)$$

Both sets (14) and (15) will be simultaneous differential equations of the second order, and their integration will result in furnishing k independent coordinates or parameters in terms of time t. There will be $2k$ arbitrary constants. The general order of procedure will be precisely the same as before: (1) Selecting k independent coordinates, such, that the coordinates x, y, z, etc., *can* be expressed in terms of them, in connection with the constraints. (2) Formation of the expression of kinetic energy and replacing in it the old variables by their new equivalents $q_1, \cdots$, etc., $q_1', \cdots$, etc. (3) Formation of U, if any; if not, computing the generalized forces Q; (4) formation of the equations (14) or (15) proper; and (5) integration of the latter. For the reason outlined above the integral of kinetic energy, if it exists, can replace any (generally the most complicated) Lagrange's equation.

A few examples will now be given; the reader is advised not to skip any of them as "uninteresting" or "useless." They have been selected with great care and are being presented in a certain order, which is designed to make matters absolutely clear.

Example 1. *A bug, of mass m, can crawl with a given velocity upon a wire* (fig. 34) *of the length* $2a$ *and mass m, the*

ends of the wire sliding upon a circumference (in a horizontal plane) of radius R. Form the equations of motion and calculate the angle through which the wire will have moved when the bug reaches the end B.

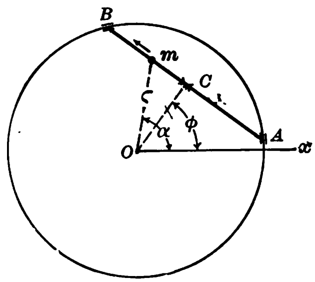


FIG. 34.

The notations will be as follows: the angle COx will be φ, and the distance $mC = r$. The bug starts from C and its velocity, $v = r/t$, is constant; also let $Om = \rho$, and $mOx = \alpha$ (polar coordinates of m). Here we have only one parameter or independent coordinate, the angle φ, to be found in terms of t, because the position of the bug upon the wire is always known for any time, t, through its velocity, so that $r = vt$. The kinetic energy of the system will be the sum of kinetic energy of the bug $(mv^2/2)$, moving forward, and of the wire, rotating backward, about its instantaneous center $O[= I(\omega^2/2)$, where I is the moment of inertia of the wire about O and is equal to mk^2, k being the proper radius of gyration; of course ω will be simply $= d\varphi/dt]$, so that the kinetic energy of the wire will be $= (mk^2/2)\varphi'^2$; that of the bug will be $(m/2)(\rho'^2 + \rho^2\alpha'^2)$ (from the ordinary expression of velocity in polar coordinates, ρ and α).

Hence $2T = mk^2\varphi'^2 + m(\rho'^2 + \rho^2\alpha'^2)$. We shall now eliminate the angle α; since $\rho = \sqrt{R^2 - a^2 + v^2t^2}$, it will readily be seen that $\rho' = v^2t/\rho$; also

$$\alpha' = \varphi' + \frac{v\sqrt{R^2 - a^2}}{\rho^2}$$

$$\left(\text{from } \alpha = \varphi + \tan^{-1}\frac{vt}{\sqrt{R^2 - a^2}}\text{; see triangle } mOC\right)$$

Finally

$$2T = mk^2\varphi'^2 + \frac{mv^4t^2}{\rho^2} + m\left(\rho\varphi' + \frac{v\sqrt{R^2 - a^2}}{\rho}\right)^2$$

where ρ is a function of time only. There being no external forces, we shall have as our only Lagrange's equation, for the parameter φ,

$$\frac{d}{dt}\frac{\partial T}{\partial \varphi'} - \frac{\partial T}{\partial \varphi} = 0;$$

that is

$$\frac{d}{dt}[mk^2\varphi' + m(\rho^2\varphi' + v\sqrt{R^2 - a^2})] = 0,$$

whence $k^2\varphi' + \rho^2\varphi' + v\sqrt{R^2 - a^2} = \text{const.} = c$.

From the initial conditions (starting from rest) it follows that $c = 0$. From the equation just obtained we can find

$$\varphi' = \frac{d\varphi}{dt} = -\frac{v\sqrt{R^2 - a^2}}{k^2 + \rho^2} = -\frac{v\sqrt{R^2 - a^2}}{k^2 + R^2 - a^2 + v^2t^2}$$

or, integrating,

$$\varphi = \varphi_0 - \sqrt{\frac{R^2 - a^2}{k^2 + R^2 - a^2}}\tan^{-1}\frac{vt}{\sqrt{k^2 + R^2 - a^2}},$$

where φ_0 is the initial value of φ. When the bug reaches B we have $vt = a$, which in the above equation gives the final value of φ.

Example 2. *A particle of mass m* (fig. 35) *is placed inside of a thin hollow ring of mass M, which can roll, without sliding, and in a vertical plane, upon a horizontal line; no friction; no initial velocities. Find the motion of this system.*

Let O be the initial point of contact, so that originally A was at O; in the time t the contact is at P, so that the distance $PO =$ the arc $AP = r\theta$, where θ is the angle corresponding to the arc; also let φ be the angle, characterizing the present position of the particle m (which originally was higher, say at

m_1). The only independent coordinates we need are the angles θ and φ, since they completely identify the system (as soon

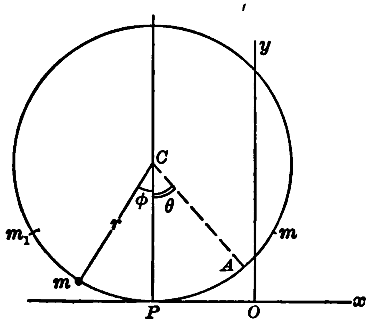


FIG. 35.

as they are found in terms of t). The coordinates of m (referred to the old axes, x, y) are,

$$x = r(\theta + \sin\varphi); \qquad y = r(1 - \cos\varphi);$$

the kinetic energy of the particle, in its own motion inside the ring, is

$$\frac{mv^2}{2} = \frac{m}{2}(x'^2 + y'^2) = \frac{mr^2}{2}(\theta'^2 + \varphi'^2 + 2\theta'\varphi' \cos\varphi)$$

(note that here again absolute coordinates have been used, although the rules of relative motion would have resulted in precisely the same expression; compare with the end of Chap. II).

The kinetic energy of the ring proper will be based upon its instantaneous rotation about the point of contact, P (but *not* about C, which itself is in motion); the instantaneous value of the angular velocity about P is $d\theta/dt$ or θ'. The moment of inertia about P will be $2Mr^2$ (compare with Bowser, Anal. Mech., example 9, p. 448, where $a = b$); hence the total kinetic energy

$$2T = mr^2(\theta'^2 + \varphi'^2 + 2\theta'\varphi' \cos\varphi) + 2Mr^2\theta'^2.$$

The only acting force being gravity, we have a potential function, $U = -mgy = -mgr(1 - \cos\varphi)$ (—, because it acts against positive y).

We now have everything necessary for forming Lagrange's equations. Taking partial derivatives of T with respect to θ and φ; also to their derivatives θ' and φ', we have

$$\frac{\partial T}{\partial \theta'} = mr^2(\theta' + \varphi' \cos\varphi) + 2Mr^2\theta'; \qquad \frac{\partial T}{\partial \theta} = 0;$$

$$\frac{\partial T}{\partial \varphi'} = mr(\varphi' + \theta' \cos\varphi); \qquad \frac{\partial T}{\partial \varphi} = -mr^2\theta'\varphi' \sin\varphi$$

also

$$\frac{\partial U}{\partial \theta} = 0; \qquad \frac{\partial U}{\partial \varphi} = -mgr \sin\varphi;$$

hence Lagrange's equations are

$$\frac{d}{dt}[m(\theta' + \varphi' \cos\varphi) + 2M\theta'] = 0, \tag{1}$$

$$mr\left(\frac{d\varphi'}{dt} + \cos\varphi \frac{d\theta'}{dt}\right) + mg \sin\varphi = 0. \tag{2}$$

Integrating (1), we have $m(\theta' + \varphi' \cos\varphi) + 2M\theta' = 0$ (the constant of integration is $= 0$ because of the initial conditions: starting from rest means $\varphi_0' = 0$; $\theta_0' = 0$). Integrating again,

$$m(\theta + \sin\varphi + 2M\theta) = m \sin\alpha$$

(where α is the initial value of φ when the particle was at rest at m_1; whence

$$\theta = \frac{m}{2M + m}(\sin\alpha - \sin\varphi). \tag{3}$$

On the other hand the equation (2) gives

$$\frac{d\varphi'}{dt} + \cos\varphi \frac{d\theta'}{dt} = -\frac{g}{r} \sin\varphi;$$

substituting $d\theta'/dt$ (derived from (3)), multiplying by $d\varphi/dt$ and integrating, we have

$$\varphi'^2 - \frac{m}{2M+m}\varphi'^2\cos^2\varphi = \frac{2g}{r}(\cos\varphi - \cos\alpha)$$

(check back by differentiating this with respect to t); whence

$$\frac{d\varphi}{dt} = \pm\sqrt{\frac{2g(2M+m)}{r}}\sqrt{\frac{\cos\varphi - \cos\alpha}{2M + m\sin^2\varphi}}. \tag{4}$$

This angular velocity, which was evidently equal to 0 when φ was $= \alpha$ (in the beginning of motion) will again be equal to 0, when $\varphi = -\alpha$; that is the particle will reach the same height on the other side of the contact point, at m_2, and will oscillate between these two points. Meanwhile [see (3)] the angle θ will have reached a certain maximum (when $\varphi = -\alpha$) $= (2m\sin\alpha)/(2M+m)$, and will be back to 0 when φ is at m_1 again; of course y oscillates between $r(1-\cos\alpha)$ and 0; while x will vary from

$$r\sin\alpha \qquad \text{to} \qquad \frac{m-2M}{m+2M}r\sin\alpha.$$

Example 3. *Two rods, A and B* (fig. 36), *connected at A*

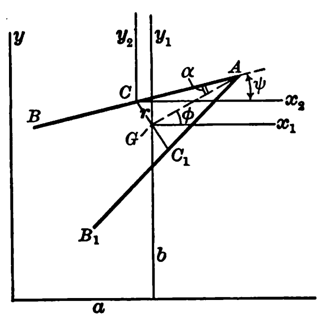


FIG. 36.

by a flexible joint, are placed upon a frictionless horizontal plane. The length of each rod is $= 2l$, *and the mass of each* $= M$. *Find the general equations of motion.*

In selecting the independent coordinates for this case we are confronted with the necessity of having not less than four such parameters: two coordinates, a and b, of the center of gravity G of the system, which is half-way between the middle points of each rod, C and C_1: one angular parameter φ, the angle of the line G–A with the axis of x; and another angular parameter α, = half the angle between the rods. We can readily satisfy ourselves as to the possibility of reconstructing the system from these four parameters: two first coordinates locate the center, G; then the line G–A is drawn and the length $l \cos \alpha$ is laid off upon it; this gives the location of the joint A; finally the rods are laid off at the angle α on both sides of G–A.

The kinetic energy will consist of two parts (Koenig's theorem): (1) kinetic energy due to the whole mass $2M$, considered as concentrated at the center of gravity, G, which equals

$$2M\frac{v^2}{2} = M(a'^2 + b'^2);$$

and (2) kinetic energy due to the motion about the center of gravity; this kinetic energy can likewise be split into two separate items, for each rod: the part due to the motion of C, the center of gravity of the rod, with the whole mass M concentrated in it $= Mv^2/2$; or, in polar coordinates, r, θ (where θ is $= x_1GC = \varphi + (\pi/2)$), this is

$$= \frac{M}{2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2\right]$$

or, substituting $r = GC = l \sin \alpha$, we have

$$\frac{M}{2}\,(l^2\alpha'^2\cos^2\alpha + l^2\varphi'^2\sin^2\alpha).$$

The other part is due to the instantaneous rotation of AB about its own center of gravity, C, the mass of the rod being

M (and the radius of gyration $= k$) and the angular velocity being $= (d\psi/dt)$, where ψ is the changing angle x_2CA, which $= \varphi - \alpha$. This part of the kinetic energy will then be

$$M\frac{k^2}{2}\left(\frac{d\psi}{dt}\right)^2 = \frac{Mk^2}{2}(\varphi' - \alpha')^2;$$

The corresponding expression for the other rod, AB', will be found by substituting $-\alpha$ for α; so that the total kinetic energy will be

$$2T = M[a'^2 + b'^2 + (l^2\cos^2\alpha + k^2)\alpha'^2 + (l^2\sin^2\alpha + k^2)\varphi'^2].$$

The mechanism of finding this kinetic energy is extremely instructive and should be thoroughly mastered by the reader). There are no external forces, therefore $U = 0$ and all Q will be $= 0$.

Writing down Lagrange's equations for a we have

$$\frac{da'}{dt} = 0, \qquad \text{or} \qquad a' = \text{const.};$$

in the same manner $b' = \text{const.}$, so that the motion of the center of gravity is rectilinear and uniform.

The equation for φ will be

$$\frac{d}{dt}\frac{\partial T}{\partial \varphi'} - \frac{\partial T}{\partial \varphi} = 0;$$

whence (since T does not contain φ)

$$\frac{d}{dt}\frac{\partial T}{\partial \varphi'} = 0; \qquad \text{or} \qquad \frac{\partial T}{\partial \varphi'} = \text{const.}; \qquad \text{or} \tag{1}$$

$$(l^2\sin^2\alpha + k^2)\varphi' = C.$$

Lagrange's equation for α would be too complicated and we shall replace it by the integral of kinetic energy $T = U + h$, or, in our case, $T = \text{const.}$; that is

$$(l^2\cos^2\alpha + k^2)\alpha'^2 + (l^2\sin^2\alpha + k^2)\varphi'^2 = A^2, \tag{2}$$

where A is a constant, absorbing also a'^2 and b'^2, both of which are constants. From (1) we see that φ' has a constant sign, so that the line G–A is rotating in the same direction about G, the angular velocity having the limits C/k^2 and $C/(l^2 + k^2)$. Substituting θ' from (1) in (2) we have

$$\alpha'^2(l^2 \cos^2 \alpha + k^2)(l^2 \sin^2 \alpha + k^2) = A^2l^2 \sin^2 \alpha + A^2k^2 - c^2,$$

of which since the first member is positive, the second member must be positive as well. Three assumptions can be made: (*a*) If $C^2 - A^2k^2$ is negative, the value of α may be any, and the angle between the rods will (according as α' is $+$ or $-$), increase or decrease until they come in contact (that is $\alpha = 0$, or $\alpha = \pi$); (*b*) if $C^2 - A^2k^2$ is positive, we can always select a constant β, such that $C^2 - A^2k^2 = A^2l^2 \sin^2 \beta$; indeed $C^2 - A^2k^2$ is always less than A^2l^2, since at the start $\alpha = \alpha_0$, and regardless of the value of α_0 we always have $A^2l^2 \sin \alpha_0 > C^2 - A^2k^2$. Therefore, in this case, the only condition for α will be $\sin^2 \alpha > \sin^2 \beta$; so that α can take any value between β and $\pi - \beta$, and the rods will oscillate in relation to G–A; (*c*) if $C^2 - A^2k^2 = 0$, α can take any value, but the less the value $\sin \alpha$, the slower the angular velocity α'; the rods will therefore tend to close either way ($\alpha = 0$, or $\alpha = \pi$) without ever reaching that state.

This discussion is due to M. Appell and is especially instructive.

Example 4. *A system* (fig. 37) *consists of two rods A–B and C–D, each of mass p and of length 2a; also of two rods, A–D and B–C, each of mass q and of length 2b; the flexible joints at the corners are frictionless. Find the motion of this system.*

Here we need but two independent coordinates, the angles θ and φ, of the rods A–B and A–D with the vertical. The kinetic energy of A–B will be due simply to its rotation about O; the moment of inertia being $= (ml^2/12)$ (Bowser, Anal. Mech., p. 431); we have

$$\text{K. E.}_{AB} = \tfrac{1}{6}pa^2\theta'^2.$$

To find that of D–C we must add to the above (which is the same as the kinetic energy of D–C about its own center), the

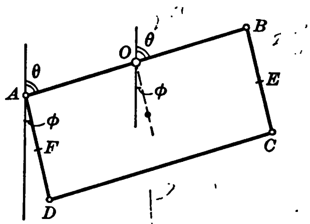


FIG. 37.

kinetic energy due to the motion of its center about O; and that is $= (p/2)AD^2 \cdot \varphi'^2 = 2pb^2\varphi'^2$; so that

$$\text{K. E.}_{CD} = \tfrac{1}{6}pa^2\theta'^2 + 2pb^2\varphi'^2;$$

In order to find the kinetic energy of A–D and B–C we must add the double kinetic energy $\frac{1}{2}(\frac{2}{3}qb^2\varphi'^2)$, due to their instantaneous rotation φ' about their centers of gravity E and F, to the double kinetic energy due to the mass q of each, concentrated at E and F, in the motion of these points, that is

$$\tfrac{1}{2}[2q(b^2\varphi'^2 + a^2\theta'^2)].$$

(Remark: This is quite evident; according to Koenig's theorem we first calculate the kinetic energy of F in its motion about A, which is $qb^2\varphi'^2$; and then add the kinetic energy due to q, concentrated in A, in its motion about O.) Finally the total kinetic energy is

$$2T = 2p(2b^2\varphi'^2 + \tfrac{1}{3}a^2\theta'^2) + 2q(a^2\theta'^2 + \tfrac{4}{3}b^2\varphi'^2).$$

We have a potential function U, equal to the mass times the distance of the center of gravity of the whole system under O; the mass being $2p + 2q$ we have,

$$U = 2(p + q)gb \cos \varphi.$$

Forming, now, the two Lagrange's equations in θ and φ we

have

$$\frac{\partial T}{\partial \theta'} = \frac{2p + 6q}{3} a^2\theta'; \qquad \frac{\partial T}{\partial \varphi'} = \frac{4(3p + 2q)}{3} b^2\varphi'^2;$$

$$\frac{\partial T}{\partial \theta} = 0; \qquad \frac{\partial T}{\partial \varphi} = 0;$$

$$\frac{\partial U}{\partial \theta} = 0; \qquad \frac{\partial U}{\partial \varphi} = -2(p + q)gb \sin \varphi.$$

Hence the equations

$$\frac{d}{dt}\left(\frac{2p + 6q}{3} a^2\theta'\right) = 0,$$

$$\frac{d}{dt}\left(\tfrac{4}{3}(3p + 2q)b^2\varphi'\right) = -2(p + q)gb \sin \varphi.$$

From the first equation we can conclude that θ' is constant, so that the rods A–B and C–D turn with a constant angular velocity. From the second equation we have

$$\frac{d^2\varphi}{dt^2} = -\frac{3}{2}\,\frac{p + q}{3p + 2q}\,\frac{g}{b} \sin \varphi;$$

comparing this with Bowser, Anal. Mech., p. 458, we conclude that the motion will be pendular, equivalent to a simple pendulum of length

$$\frac{6p + 4q}{3p + 3q} b.$$

Remark: We already had equations of this sort on two occasions; it is well to commit to memory the fact that

$$\frac{d^2\varphi}{dt^2} = -\frac{g}{a} \sin \varphi$$

means pendular motion, equivalent to a simple pendulum of length a.

A few examples will now be given on the motion of rigid bodies; their position is generally given by means of Euler's

angles, locating the principal axes; the reader should, therefore, revise the end of the first chapter.

Example 5. *A body of revolution is mounted on a straight wire R–S* (fig. 38), *coinciding with its axis, one end of which*

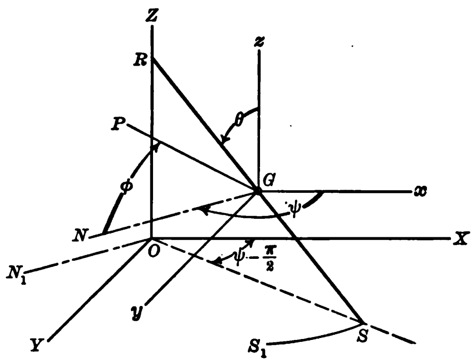


FIG. 38.

can slide along a vertical O–Z, while the lower end, S, can slide in any way on the horizontal plane x–O–y; no friction and no external forces except gravity. Find the motion.

Three angular parameters are necessary to define the position of the body: two will locate the wire and the third parameter will define the body in its possible rotation around (and including) the wire.

The body is not shown on the sketch; its center of gravity is at G; through G draw axes x, y, z, parallel to X, Y, Z, and let N–G be the intersection of a plane, perpendicular to the wire R–S with the plane x, G, y. Call ψ the angle xGN; θ the angle RGz; and φ the angle between GN and any line GP in the plane perpendicular to R–S and fixed in the body. Then, the angle φ will completely locate the body about R–S; also $ORS = \theta$, thus fixing the position of the wire about O–z. On the other hand, the angle NGS which is a right angle, projects as a right angle upon the plane x–O–y (since one of its sides, NG, is parallel to that plane); hence $XOS = \psi$

$-(\pi/2)$, which fixes the plane ORS in relation to XOZ. These three angles ψ, θ and φ, are Euler's angles and the position of the system is clearly identified by them. If given the angles ψ, θ and φ we could proceed to locate the system as follows: From the angle ψ we can find $\psi - (\pi/2)$, and thereby locate the plane ROS; the angle θ will definitely locate the wire in that plane; finally, the position of the body, in whatever rotation it might have about R–S, will be identified by any such line as PG (belonging to the body), through the angle φ, from the intersection NG. The motion which we are investigating is perfectly general; the end R may move down along Z; the end S may, at the same time, describe some arc S–S_1, and, meanwhile, the body itself may turn together with its axis about R–S. Whether all these motions will or will not take place, in this most general case, will be seen from the equations of motion; at any rate the motion is perfectly defined by these Euler's angles; they will be our independent coordinates.

The next step is to find the kinetic energy of such a system: this will (Koenig's theorem) consist of two parts: (1) The kinetic energy due to the motion of the center of gravity, G, with the whole mass, M, concentrated in it; also, (2) the kinetic energy due to the motion of the body about the center of gravity, considered as fixed. Let us calculate these items separately:

1. Let the distance $RG = r$, and $RS = l$; the coordinates x, y, z, of G, are as is very clearly shown on the sketch

$$x = r\sin\theta\sin\psi; \qquad y = r\sin\theta\cos\psi; \qquad z = (l - r)\cos\varphi.$$

Differentiating with respect to t and substituting into

$$2T = M(x'^2 + y'^2 + z'^2),$$

we have

$$2T = M[r^2(\sin^2\theta\cdot\psi'^2 + \cos^2\theta\cdot\theta'^2) + (l - r)^2\sin^2\theta\cdot\theta'^2].$$

2. The kinetic energy of the body about its center of gravity can be due only to its (possible) rotation about G, considered as a fixed point. We had a general expression of kinetic energy for such a case (see under *Body with one point fixed*)

$$2T = A\omega_1^2 + B\omega_2^2 + C\omega_3^2;$$

where A, B and C are the moments of inertia about the principal axes of the body and ω_1, ω_2, ω_3, are the components, of the instantaneous velocity of rotation, on some initial axes, in relation to which the principal axes are given by Euler's angles. We do not care to have the instantaneous velocity itself, but we had the following expressions for ω_1, ω_2, ω_3 in terms of Euler's angles

$$\omega_1 = \sin\theta \sin\varphi\cdot\psi + \cos\varphi\cdot\theta',$$
$$\omega_2 = \sin\theta \cos\varphi\cdot\psi' - \sin\varphi\cdot\theta',$$
$$\omega_3 = \cos\theta\cdot\psi' + \varphi'$$

(see under Euler's angles, at the end of chapter 1). In bodies of revolution it is customary to denote the two equal moments of inertia by A and B; in our case $A = B$ are the moments of inertia about any two principal axes at right angles to each other, in the plane, perpendicular to $R - S$; while the moment of inertia about $R - S$ will be $= C$. Therefore in our case

$$2T = A(\omega_1^2 + \omega_2^2) + C\omega_3^2;$$

which, after substitution of ω_1, ω_2, ω_3, just given, becomes

$$2T = A(\sin^2\theta\cdot\psi'^2 + \theta'^2) + C(\cos\theta\cdot\psi' + \varphi')^2$$

so that the total kinetic energy of the moving body is

$$2T = (A + Mr^2)\sin^2\theta\cdot\psi'^2 + [A + M(r^2\cos^2\theta$$
$$+ (l - r)^2\sin^2\theta]\theta'^2 + C(\cos\theta_1\cdot\psi' + \varphi')^2.$$

The only external force is gravity, deriving from the potential function $U = Mgz$, that is $= Mg(l - r)\cos$

In forming Lagrange's equations we can see at once that both T and U contain no φ or ψ; therefore we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \varphi'}\right) = 0; \qquad \text{and} \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \psi'}\right) = 0;$$

that is

$$\frac{\partial T}{\partial \varphi'} = \text{const.}; \qquad \text{or} \qquad \varphi' + \psi' \cos\theta = \text{const.} = a; \quad (1)$$

also

$$\frac{\partial T}{\partial \psi'} = \text{const.}; \qquad \text{or}$$

$$(A + Mr^2)\sin^2\theta\cdot\psi' + C\cos\theta(\varphi' + \psi'\cos\theta) = \text{const.};$$
$$\text{or} \qquad \psi'\sin^2\theta + m\cos\theta = b \quad (2)$$

$$\left[\text{where } m = \frac{aC}{(A + Mr^2)}\right]$$

Instead of the third Lagrange's equation (in θ) we can substitute the integral of kinetic energy, $T = U + h$, that is,

$$\psi'^2\sin^2\theta + (1 + n\sin^2\theta)\theta'^2 = p\cos\theta + h \quad (3)$$

(where

$$n = \frac{Ml(l - 2r)}{A + Mr^2} \qquad \text{and} \qquad p = \frac{2Mg(l - r)}{A + Mr^2},$$

eliminating ψ between (2) and (3) we shall have $t = \int F(\theta)d\theta$, where $F(\theta)$ is a function of θ

$$F(\theta) = \sqrt{\frac{1 + n\sin^2\theta}{(p\cos\theta + h)\sin^2\theta - (b - m\cos\theta)^2}} \times \sin\theta;$$

therefore, from (2) we have

$$\psi = \int \frac{b - m\cos\theta}{\sin^2\theta} F(\theta)d\theta$$

and finally

$$\varphi = \int \left(a - \frac{\cos\theta(b - m\cos\theta)}{\sin^2\theta} \right) F(\theta)d\theta,$$

so that the problem is reduced to integration.

This example has been given here as an illustration of the mechanism of the introduction of Euler's angles; for this reason we shall omit the interesting discussion given in M. Painlevé's lectures, from which the example has been taken.

Example 6. *A solid of revolution, having one point of its axis fixed at O, and whose initial rotation about its axis is given, is subject only to the external force of gravity. Investigate the motion of such a top or gyroscope.*

The position of a rigid body of this sort can be given at any time through Euler's angles ψ, θ, φ, which we shall take for independent coordinates, the axis of spin being z. Let (fig. 39) G be the center of gravity and let $OG = l$; the poten-

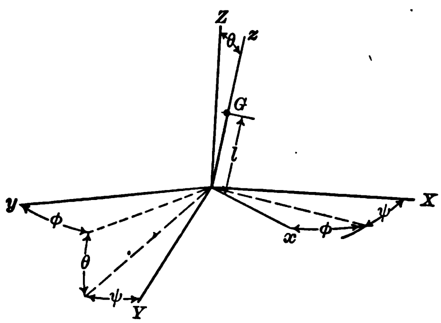


FIG. 39.

tial function will be $U = Mgl\cos\theta$ and the expression of kinetic energy, $2T = A\omega_1^2 + B\omega_2^2 + C\omega_3^2$, will be given by the formulae (see example 5)

$$\omega_1 = \sin\theta\sin\varphi\cdot\psi' + \cos\varphi\cdot\theta';$$

$$\omega_2 = \sin\theta\cos\varphi\cdot\psi' - \sin\varphi\cdot\theta';$$

$$\omega_3 = \cos\theta\cdot\varphi' + \varphi'$$

from which, considering that here, too, we have equal moments of inertia, $A = B$, we have

$$2T = A(\theta'^2 + \psi'^2 \sin^2 \theta) + C(\varphi' + \psi' \cos \theta)^2.$$

The three Lagrange's equations, in φ, ψ and θ are

$$C\frac{d}{dt}(\varphi' + \psi' \cos \theta) = 0,$$

$$\frac{d}{dt}(A\psi' \sin^2 \theta + C(\varphi' + \psi' \cos \theta) \cos \theta) = 0,$$

$$A\frac{d\theta'}{dt} - A\psi'^2 \sin \theta \cos \theta + C(\varphi' + \psi' \cos \theta)\psi' \sin \theta = - Mgl \sin \theta;$$

from the first two we have

$$\varphi' + \psi' \cos \theta = n; \tag{1}$$

$$A\psi' \sin^2 \theta + Cn \cos \theta = k; \tag{2}$$

n and k being constants; the third equation gives

$$A\frac{d\theta'}{dt} - A\psi'^2 \sin \theta \cos \theta + Cn\psi' \sin \theta = - Mgl \sin \theta. \tag{3}$$

Eliminating ψ' from the last two equations we can obtain a rather complicated expression of θ in terms of t.

But we can investigate these results in an elementary manner, and derive therefrom a few fundamental principles of gyroscopic motion.

We shall begin by replacing (3) by its equivalent, the integral of kinetic energy $T = U + h$; in our case

$$2T = A(\theta'^2 + \psi'^2 \sin \theta) + C(\varphi' + \psi' \cos \theta)^2$$

and

$$U = Mgl \cos \theta \quad \text{(see also (1))},$$

and therefore the integral of kinetic energy will be

$$A(\theta'^2 + \psi'^2 \sin^2 \theta) + Cn^2 - 2mgl \cos \theta = 2h, \tag{4}$$

from which ψ' can be eliminated through (2); so that

$$A\theta'^2 + \frac{(k - Cn\cos\theta)^2}{A\sin^2\theta} + Cn^2 - 2mgl\cos\theta = 2h. \quad (5)$$

Now in the equation (2), let α be the initial value of θ, that is let $\theta = \alpha$ when $t = 0$ and when $\psi' = 0$; from this substitution we have

$$k = Cn\cos\alpha \quad \text{and} \quad A\psi'\sin^2\theta = Cn(\cos\alpha - \cos\theta). \quad (6)$$

Also k substituted in (5) results in

$$A\theta'^2 + C^2\frac{n^2(\cos\alpha - \cos\theta)^2}{A\sin^2\theta} + Cn^2 - 2mgl\cos\theta = 2h;$$

so that, in the beginning of motion, when $\theta = \alpha$ and $\theta' = 0$, we have

$$Cn^2 - 2mgl\cos\theta = 2h. \quad (7)$$

This substituted in (5) gives

$$(\cos\theta - \cos\alpha)[2mglA(1 - \cos^2\theta) + C^2n^2(\cos\theta - \cos\alpha)] = 0.$$

This is a cubic equation in $\cos\theta$, one of the roots being, of course, $\cos\theta = \cos\alpha$; and the other two roots can be obained from the quadratic

$$(\cos^2\theta - 1) - \frac{C^2n^2}{2mglA}(\cos\theta - \cos\alpha) = 0;$$

here we can put

$$\frac{C^2n^2}{2mglA} = 2e,$$

so that

$$(\cos^2\theta - 1) - 2e(\cos\theta - \cos\alpha) = 0;$$

hence

$$\cos\theta = e \pm \sqrt{1 + e^2 - 2e\cos\alpha}.$$

The greater root will always be > 1 ($\cos\alpha$ being always < 1; if $\cos\alpha = 1$, then only can $\cos\theta$ be $= 1$), and therefore should be rejected, since of course $\cos\theta$ should be < 1.

Therefore the only solution is

$$\cos\theta_1 = e - \sqrt{1 + e^2 - 2e\cos\alpha}. \tag{8}$$

This shows that the value of θ (and therefore the axis of the top) will oscillate between $\theta = \alpha$ and $\theta = \theta_1$, just found.

If the rotation of spin is high, we can develop the radical and neglect powers of higher than second, so that the simplified expression will be

$$\cos\theta = \cos\alpha - \frac{1}{e}\sin^2\alpha;$$

that is to say, the limits of θ are practically α and

$$\cos^{-1}\left(\cos\alpha - \frac{\sin^2\alpha}{e}\right). \tag{9}$$

Precession. From (7) and (4) we can easily see that

$$A\theta'^2 + A\psi'^2\sin^2\theta + 2mgl(\cos\alpha - \cos\theta) = 0, \tag{10}$$

where, either $\cos\alpha - \cos\theta = 0$, or $\neq 0$.

1. If $\cos\alpha - \cos\theta$ is not $= 0$, then ψ' has a maximum whenever $\theta'^2 = 0$ in that case

$$\psi'^2 = \frac{2mgl}{A}\,\frac{\cos\alpha - \cos\theta}{\sin^2\theta},$$

or, substituting

$$\frac{A\sin^2\theta}{\cos\alpha - \cos\theta} = \frac{Cn}{\psi'}$$

from (6) we have

$$\psi' = \frac{2mgl}{Cn};$$

this is the value of the angular velocity of the line X_1 about O–Z (see Euler's angles on the sketch), or, in other words, the angular velocity with which the line O–z describes a cone about O–Z. This is not a constant but an instantaneous value, at the moment when the angle θ is in its furthest position

from its initial value α and is reversing ($\theta' = 0$) the sense of its oscillation.

2. When the axis reaches its initial value, we have $\cos\alpha = \cos\theta$, and therefore we can conclude (from 10) that at that instant both θ' and $\psi' = 0$; in other words at that moment the axis of the top is temporarily stopped in its double motion (*away from* and *about* the axis O–Z). Therefore the angular (azimuthal) velocity of the axis of the top about the vertical changes from O to $2mgl/Cn$, while, at the same time, the inclination of the axis to the vertical changes from the initial value α to its other value, θ_1, which we had derived above (8). Therefore the motion of the top on its axis will project upon the horizontal plane in a manner shown in fig. 40, where the

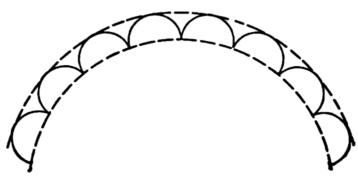


FIG. 40.

inner circle corresponds to the initial value $\theta = \alpha$, while the outer circle represents the extreme value of θ ($= \theta_1$).

The azimuthal motion about the vertical is called *precession;* we shall presently show that its average value is mgl/Cn; this is rather slow, since n is always considerable. The difference between the angles θ_1 and α is almost imperceptible (in view of (9) the angle varies between $\cos\alpha$ and $\cos\alpha - (2mglA\sin^2\alpha/C^2n^2)$, where n is generally very great) and the oscillation of the axis of the top between these values (so-called *nutation*) is of very short period and can but rarely be observed.

Steady motion. If we assume that the variation of the angle θ (nutation) is so slight as to be negligible, in other words that $\theta = \text{const.}$, and, consequently, that $\theta' = 0$, we have, from (3), dropping $d\theta'/dt$ and neglecting ψ'^2, (since ψ'

is slow),

$$\psi' = \frac{mgl}{Cn},$$

which is half the maximum value, derived above.

This equation can be arranged in a slightly different form, in which it is extensively used in gyroscopic work: multiplying by $\sin\theta$ we have $mgl \sin\theta = Cn\psi' \sin\theta$; the left member is the external moment, M, whose arm is l; the right member is ψ' projected upon a direction at right angles to the axis of spin of the gyroscope, and to the axis of the external moment (fig. 41). Using notations customary in

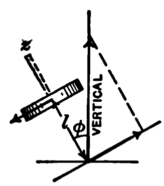


FIG. 41.

gyroscopic practice we have $M = I\Omega\omega$, where I is $= C$, that is, the moment of inertia of the gyroscope; Ω is the velocity of spin, and ω, that of the precession about an axis at right angles to the axes of both the spin and of the external moment. This formula enables us to calculate the moment necessary to create a certain precession; also, the moment that will result from giving the so-called *forced precession* to a spinning body. It can also be put in the following form

$$M = \frac{WR^2Nn}{2937},$$

where M is the resulting moment in ft.-lbs.; W the weight of the spinning body of lbs.; R is the radius of gyration in ft. and N and n are the velocities, of spin and of precession, both in revolutions per minute.

Remark. The reader will now understand the significance

of the constant n (see equation (1)). It is not the velocity of spin; the latter is $= \varphi'$, and forms only a part of n; the other part, $\psi' \cos\theta$, is the component (upon the axis of z) of the precession, ψ'; on the other hand ψ' is never very great and therefore the value of n does not differ materially from the velocity of spin.

Example 7. *To derive Euler's equations from Lagrange's equations.*

The expression of kinetic energy for a body with one point fixed is

$$2T = A\omega_1^2 + B\omega_2^2 + C\omega_3^2, \tag{1}$$

where ω_1, ω_2, ω_3 are the components of the instantaneous rotation, ω, upon the principal axes (strictly speaking upon fixed directions, temporarily coinciding with the latter). These components can be given in terms of Euler's angles by the transformation formulae which we had before

$$\begin{aligned} \omega_1 &= \psi' \sin\theta \sin\varphi + \theta' \cos\varphi, \\ \omega_2 &= \psi' \sin\theta \cos\varphi - \theta' \sin\varphi, \\ \omega_3 &= \psi' \cos\theta + \varphi', \end{aligned} \tag{2}$$

where Euler's angles, ψ, θ, φ, identify the principal axes of the system, taken about the fixed point. Taking the angles ψ, θ, φ as independent coordinates and putting virtual work $\delta W = \Psi\delta\psi + \Theta\delta\theta + \Phi\delta\varphi$ (where Ψ, Θ and Φ are generalized forces, in our case *moments*) we can write down one of Lagrange's equations, in φ, as follows

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \varphi'}\right) - \frac{\partial T}{\partial \varphi} = \Phi.$$

Now, in order to derive $\partial T/\partial\varphi'$, we might substitute (2) into (1) and differentiate the result with respect to φ'; but this is not necessary, since, from (1) and the last equation (2) we have

$$\frac{\partial T}{\partial \varphi_1} = \frac{\partial T}{\partial \omega_3}\frac{\partial \omega_3}{\partial \varphi_1} = C\omega_3$$

and also

$$\frac{\partial T}{\partial \varphi} = \frac{\partial T}{\partial \omega_1}\frac{\partial \omega_1}{\partial \varphi} + \frac{\partial T}{\partial \omega_2}\frac{\partial \omega_2}{\partial \varphi} = A\omega_1\frac{\partial \omega_1}{\partial \varphi} + B\omega_2\frac{\partial \omega_2}{\partial \varphi}.$$

Differentiating ω_1 and ω_2 (transformation formulae), we have

$$\frac{\partial \omega_1}{\partial \varphi} = -\theta' \sin\varphi + \psi' \cos\varphi \sin\theta = \omega_2, \qquad \frac{\partial \omega_2}{\partial \varphi} = -\omega_1,$$

Therefore

$$\frac{\partial T}{\partial \varphi} = \omega_1\omega_2(A - B)$$

we have

$$C\frac{d\omega_3}{dt} + (B - A)\omega_1\omega_2 = \Phi.$$

That Φ, the generalized force corresponding to the generalized displacement $\delta\varphi$, is merely the external moment, N, about the axis z, can be shown as follows: Our rule requires forming of the equation

$$Q = \Sigma\left(X\frac{\partial x}{\partial q} + Y\frac{\partial y}{\partial q} + Z\frac{\partial z}{\partial q}\right)$$

for every generalized force Q we wish to determine. In this particular case Q is Φ which we want to find; and, considering that we are dealing with the rotation φ, that is about the axis z, assuming other displacements constant, we have (for rotation about z—polar coordinates $x = r\cos\varphi$; $y = r\sin\varphi$)

$$\frac{\partial x}{\partial \varphi} = -r\sin\varphi = -y; \qquad \frac{\partial y}{\partial \varphi} = r\cos\varphi = x; \qquad \frac{\partial z}{\partial \varphi} = 0;$$

and therefore

$$\Sigma\left(X\frac{\partial x}{\partial \varphi} + Y\frac{\partial y}{\partial \varphi} + Z\frac{\partial z}{\partial \varphi}\right) = \Sigma(xY - yX),$$

which is evidently the moment N about the axis z; therefore $\Phi = N$, and we have the third Euler's equation

$$C\frac{d\omega_3}{dt} + (B - A)\omega_1\omega_2 = N.$$

This equation contains no angles ψ, θ and φ, and therefore is perfectly general in regard to all axes, subject, of course, to suitable change in the notations.

Note. Holonomous and not-holonomous systems. In deriving Lagrange's equations mention has been made of the perfect freedom with which generalized coordinates can be selected. The only implied condition was that the ordinary position-coordinates, x, y, z, etc., *can* be expressed through the generalized coordinates by means of some such equations as (2) p. 89, which may or may not contain time. A very important remark should be made, however, regarding the choice of these independent coordinates. The equations (2), just mentioned, *should contain no time derivatives;* otherwise Lagrange's equations cannot be applied. In practical work the reader will but seldom meet with difficulties of this nature, and in more advanced treatises he will find how corrections should be made so that modified Lagrange's equations can be applied to such special problems. Systems which can be characterized by equations like (2) involving no time derivatives, are known as *holonomous;* those, which in the equations like (2) would contain time derivatives are called *not-holonomous* systems. For instance, in evolving Euler's equations by Lagrange's method (example 7, p. 141) what was our object in introducing the transformation formulae (2) into the expression of kinetic energy (1)? Why could not Lagrange's equations be obtained directly by selecting ω_1, ω_2, ω_3 as independent coordinates, and by differentiating (1) with respect to them? The reason lies in the fact that the coordinates of any point of a body can be expressed through ω_1, ω_2, ω_3 *only* by means of relations involving derivatives dx/dt, etc. (see p. 48). Therefore we had recourse to the transformation formulae (2) taking Euler's angles as independent coordinates, well knowing that by means of Euler's angles the system can be completely characterized by expressions *involving no derivatives* (see p. 80).

CHAPTER IV.

Lagrange's Equations for Relative Motion.

We have seen why the relative motion of a *particle* does not need any new methods. In dealing with a *system*, three separate methods can be used for forming Lagrange's equations of motion.

First method. Let it be required to find the motion of a system in relation to moving axes x, y, z, the motion of which, in reference to some fixed axes, X, Y, Z, is known. We can select independent coordinates, q_1, q_2, $\cdots$, etc., in relation to the moving system x, y, z, and then deal with the problem as if the motion were absolute, since any parameters, say q_1, q_2, $\cdots$, characterizing the system in reference to moving axes, x, y, z, will also define it in reference to the fixed set, X, Y, Z, the motion of x, y, z, themselves being known. In calculating the energy, however, the following should be remembered: The expression of kinetic energy is

$$2T = \Sigma m(x'^2 + y'^2 + z'^2),$$

where x, y, z, are absolute coordinates, which should be found in terms of q_1, q_2, $\cdots$, and then substituted, which will give $2T$ in terms of q's and q''s. A much simpler way would be to form the expression of energy by means of the formula of absolute velocity in the most general case of motion (see (2) under *Relative motion*)

$$V_{ax} = \frac{dx}{dt} + v_{marx} + \omega_2 z - \omega_3 y,$$

$$V_{ay} = \frac{dy}{dt} + v_{mary} + \omega_3 x - \omega_1 z,$$

$$V_{az} = \frac{dz}{dt} + v_{marz} + \omega_1 y - \omega_2 x,$$

where V_{ax}, etc., are the projections of the absolute velocity; dx/dt, etc., are the projections of the relative velocity on the relative axes, x, y, z (parallel to X, Y, Z); v_{marx}, etc., are the projections upon X, Y, Z, of the (mar) motion of the moving axes of reference; while ω_1, ω_2, ω_3 are the instantaneous rotations about the axes x, y, z.

Hence the final expression of kinetic energy

$$2T = \Sigma m\Bigg[\left(\frac{dx}{dt} + v_{marx} + \omega_2 z - \omega_3 y\right)^2 + \left(\frac{dy}{dt} + v_{mary} + \omega_3 x - \omega_1 z\right)^2 + \left(\frac{dz}{dt} + v_{marz} + \omega_1 y - \omega_2 x\right)^2\Bigg].$$

This expression though apparently long is easily applied in practical work. Its advantage is that it involves only relative coordinates, which are easily found in terms of q, q', etc., while all the velocities dx/dt, $\cdots$, etc., as well as v_{marx}, $\cdots$, are known. In calculating the generalized forces, Q, the same methods will be used as those given before; and if there is a force function U, then of course $Q = \partial U/\partial q$ for each q.

Example. The plane x–O–z can turn, about a vertical axis O–z, with a constant angular velocity, ω. Find the motion of a

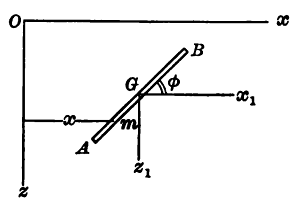


FIG. 42.

rod A–B, constrained (fig. 42), *to move in such a plane, and given any initial motion in relation to the axes x–z; no friction.*

The motion of the rod in relation to the moving axes x–O–z will be defined by three independent coordinates: a and b,

the coordinates of its center of gravity, G; and φ, the angle of the rod with, say, the axis x. The absolute velocity of any particle, m, is here made up of the relative velocity, which is in the plane x–O–z; and of the (mar) velocity, which is $= \omega x$, and is perpendicular to the moving plane at that point, m. Hence the absolute velocity $v_a^2 = v_r^2 + (v_{mar})^2$ so that the kinetic energy $2T = \Sigma mv^2 = \Sigma mv_r^2 + \Sigma mv_{mar}^2$. These two terms will be found separately. The relative motion of the rod consists of translation and of rotation; according to Koenig's theorem we have $\Sigma mv_r^2 = M(a'^2 + b'^2 + k^2\varphi'^2)$, where M is the whole mass and k the radius of gyration of A–B about G; while $\Sigma mv_{mar}^2 = \omega^2 \Sigma mx^2$; but Σmx^2 is the moment of inertia about the axis z, which as we know, equals the moment of inertia about z_1 plus mass times the square of the distance between the axes, $\Sigma mv_{mar}^2 = M\omega^2(a^2 + k^2 \cos\varphi)$. Hence finally

$$2T = M(a'^2 + b'^2 + k^2\varphi'^2 + \omega^2 k^2 \cos^2\varphi + \omega^2 a^2).$$

There exists a potential function of external forces, $U = Mgb$ so that Lagrange's equations for a, b and φ will be

$$\frac{d}{dt}(a') - \omega^2 a = 0; \qquad \frac{d}{dt}(b') = g;$$

$$\frac{d}{dt}(k^2\varphi') + k^2\omega^2 \sin\varphi \cos\varphi = 0,$$

whence a, b, φ result in terms of time, as a direct result of integrating these equations.

From the first equation we have, $a = Ae^{\omega t} + Be^{-\omega t}$ from the second, $b = g(t^2/2) + Ct + D$; and the third equation can be reduced to

$$\frac{d^2\varphi}{dt^2} + \frac{g}{m} \sin 2\varphi = 0.$$

This means pendular motion, equivalent to a simple pendulum of the length $2m$.

It will thus be seen that the first method contemplates

finding the (absolute) kinetic energy and then expressing it in relative coordinates.

Second method. According to this method we add, to the acting forces, the two additional forces (see under *Relative motion,* Chapter 1), the centrifugal force (F_{mar}), due to the motion of relative axes; and Coriolis's force, (F_c); having done so we can form the expression of kinetic energy in relation to the moving axes as if they were fixed; and, in forming the generalized forces, Q, etc., according to the general rules, we have to consider the two additional forces. An example will make this clear.

Example. Let us apply this method to the problem, for which the solution has already been given in two different manners (figs. 31 and 33).

A particle is sliding in a circular tube (fig. 43) *which itself is revolving with a constant angular velocity about a vertical axis O.*

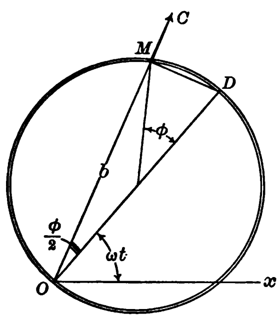


FIG. 43.

Let O–x be the reference axis, from which the rotation starts, so that the angle ωt is sufficient to define the circle by its diameter, O–D; the circle is the moving system, and only one parameter, φ, is necessary to fully locate the point M in relation to this moving system. According to the principles of the second method, all we have to do is to add to the external forces two other forces; (1) that due to the (mar) motion of the revolving system itself; and (2) the Coriolis's

force, of which the expression has been given above. There are no external forces; the centripetal acceleration v^2/r is directed radially; and so is Coriolis's acceleration, so that the only force to be considered is the force exerted on M by the rotation of the (mar), relative axes, that is of the circle itself, with the angular velocity ω. This is clearly the centrifugal force, M–C, and it equals $m\omega^2 b$, where $b = 2r \cos (\varphi/2)$.

The kinetic energy is

$$T = \frac{mv^2}{2} = \frac{m}{2} r^2 \left(\frac{d\varphi}{dt}\right)^2 = \frac{m}{2} r^2 (\varphi')^2.$$

In calculating the only generalized force, Φ, we reason as follows: What force, Φ, multiplied by the corresponding displacement, $\delta\varphi$, will give the same work as that, done by the force M–$C = m\omega^2 b$, during any possible change of b, that is δb? Of course, $b = 2r \cos (\varphi/2)$ so that

$$\delta b = - r \frac{\sin \varphi}{2} \cos \frac{\varphi}{2} \delta\varphi;$$

therefore $M - C \times \delta b = - mr^2\omega^2 \sin \varphi \cdot \delta\varphi$ or

$$\Phi\delta\varphi = - mr^2\omega^2 \sin \varphi\delta\varphi;$$

so that

$$\Phi = - mr^2\omega^2 \sin \varphi.$$

The only Lagrange's equation we have to form is (for φ):

$$\frac{d}{dt} \frac{\partial T}{\partial \varphi'} - \frac{\partial T}{\partial \varphi} = \Phi$$

hence

$$\frac{d^2\varphi}{dt^2} = - \omega^2 \sin \varphi,$$

the same as we had before.

The reader is invited to re-write this solution for a sphere of mass M and radius of gyration k (sliding on the revolving ring, instead of a particle). This will mean certain changes in the expression of T; apply Koenig's theorem.

Third method. This method is due to M. Gilbert, and is especially interesting. We know that, in order to determine the motion of a system in relation to a set of reference axes, x, y, z, which itself is in motion in relation to fixed axes, X, Y, Z, we must apply at every point of the system two additional forces: that is the (*mar*) motion, and Coriolis's force. Imagine, however, that the motion of the reference axes is referred not to fixed axes, X, Y, Z, but to an auxiliary set of axes, X_0, Y_0, Z_0 (fig. 44), which are always parallel to

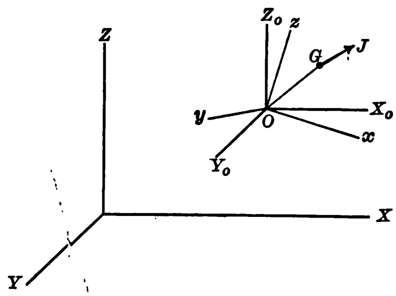


FIG. 44.

X, Y, Z. In other words, the motion of X_0, Y_0, Z_0 is *translatory* in regard to X, Y, Z; while the set x, y, z, can have any rotation about the point O; so that these two motions, translation and rotation, enable the system x, y, z, to assume any position whatever in regard to X, Y, Z; then, all we need to do is to apply, at every particle of the moving system, corresponding (*mar*) forces, after which the axes X_0, Y_0, Z_0, can be considered as fixed.

Owing to the translatory nature of the motion of X_0, Y_0, Z_0, the (*mar*) forces will be the same for every particle of the system; if the (*mar*) acceleration is J, the corresponding additional force will be $(-mJ)$. Let the projection of such forces upon the reference axes, x, y, z, be $-mJ_x$, $-mJ_y$, $-mJ_z$. To every virtual displacement, δx, δy, δz, of any particle, there will be a corresponding element of work done

by these forces, so that for the whole system this will be

$$- \Sigma m(J_x \delta x + J_y \delta y + J_z \delta z), \tag{1}$$

where J_x, J_y, J_z, are known functions of t.

If we form an expression

$$K = - \Sigma m(xJ_x + yJ_y + zJ_z),$$

which is also

$$= - [J_x \Sigma mx + J_y \Sigma my + J_z \Sigma mz]$$

and therefore

$$= - M(aJ_x + bJ_y + cJ_z), \tag{2}$$

where a, b, c, are the coordinates of the center of gravity, G, in relation to the axes x, y, z (Bowser, Anal. Mech., p. 110), we can say that (1) is $= \delta K$; it also follows that

$$K = - M \cdot J \cdot OG \cdot \cos (J, OG)$$

(from the well-known formula of Anal. Geometry: see Bowser, Anal. Geom., p. 268, formula 6).

Having thus introduced the only correction necessary (due to (*mar*) forces) we can now consider the motion as absolute. Of course, in Lagrange's equations, of the type

$$\frac{d}{dt}\frac{\partial T}{\partial q'} - \frac{\partial T}{\partial q} + = Q\ \frac{\partial K}{\partial q},$$

there will appear the last term $\partial K/\partial q$, for every q, as corresponding share of generalized forces, for any generalized displacement, δq, due to the additional function K, which we have just introduced. If the actually applied (external) forces have a potential function, we simply have the second member $= (\partial/\partial q)(U + K)$.

Regarding the kinetic energy of this motion the following remark can be made: We have seen (equation (2) under *Relative motion*), how the absolute velocity of a particle can be given through its motion in relation to the moving axes, as well as through the rotation of these axes themselves (we

can omit the terms (v_{marx}), (v_{mary}), (v_{marz}) because our motion of the reference axes is about a fixed point, O); therefore we have

$$
\begin{aligned}
V_{ax} &= \frac{dx}{dt} + \omega_2 z - \omega_3 y, \\
V_{ay} &= \frac{dy}{dt} + \omega_3 x - \omega_1 z, \\
V_{az} &= \frac{dz}{dt} + \omega_1 y - \omega_2 x.
\end{aligned} \tag{3}
$$

Substituting these expressions into the usual expression of kinetic energy

$$2T = \Sigma m(v_{ax}^2 + v_{ay}^2 + v_{az}^2),$$

we have

$$2T = \Sigma m[(x' + \omega_2 z - \omega_3 y)^2 + (y' + \omega_3 x - \omega_1 z)^2 + (z' + \omega_1 y - \omega_2 x)^2];$$

expanding and simplifying we have

$$2T = 2\boldsymbol{T} + 2\boldsymbol{G} + 2\boldsymbol{V},$$

where

$$
\begin{aligned}
\boldsymbol{T} &= \tfrac{1}{2}\Sigma m(x'^2 + y'^2 + z'^2), \\
\boldsymbol{G} &= \tfrac{1}{2}\Sigma m[(\omega_2 z - \omega_3 y)^2 + (\omega_3 x - \omega_1 z)^2 + (\omega_1 y - \omega_2 x)^2], \\
\boldsymbol{V} &= \Sigma m[x'(\omega_2 z - \omega_3 y) + y'(\omega_3 x - \omega_1 z) + z'(\omega_1 y - \omega_2 x)].
\end{aligned}
$$

These expressions can be interpreted as follows:

$\boldsymbol{T}$ is simply the kinetic energy of the system in its motion in relation to moving axes, x, y, z.

$\boldsymbol{G}$ is the kinetic energy of the system in its rotation about the instantaneous axis ω (see Euler's formulae and fig. 11) and is $= Mk^2(\omega^2/2)$, where Mk^2 is the moment of inertia about the instantaneous axis. Finally

$\boldsymbol{V}$ can be transformed as follows:

$$\boldsymbol{V} = \omega_1 \Sigma m(yz' - zy') + \omega_2 \Sigma m(zx' - xz') + \omega_3 \Sigma m(xy' - yx').$$

But we have seen that

$$\Sigma m(yz' - zy') = A\omega_1,$$
$$\Sigma m(zx' - xz') = B\omega_2,$$
$$\Sigma m(xy' - yx') = C\omega_3,$$

are the projections of the *impulse axis* (see fig. 12) P, or the *total moment of momentum;* therefore, in view of the formula of anal. geometry to which reference has just been made, we see that

$$V = \omega \cdot P \cdot \cos(\omega, P).$$

In other words, V is the product of the impulse axis, the instantaneous angular velocity and the cosine of the angle between them.

These separate items T, G and V are not difficult to find in most practical problems, as will appear from the following examples:

Example 1. *A particle of unit mass is constrained to move in a vertical plane B* (fig. 45) *which has a given rotation, ω, about a vertical axis Z; the only external force is gravity.*

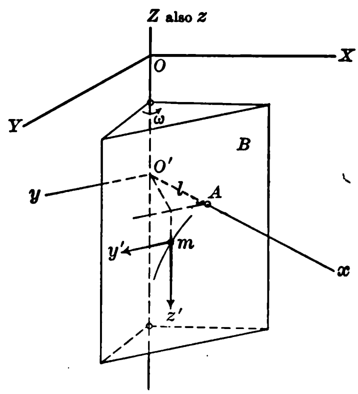


FIG. 45.

Let X, Y, Z, be the fixed axes; we shall take, as moving axes, x, y, z, where z can be directed upward, x, perpendicular to the plane, B, at any point, A; and y parallel to the plane.

The auxiliary axes, X_0, Y_0, Z_0, would be drawn through O', parallel to X and Y, but they are not shown. Let O' be anywhere on the axis Z and let $O–A = l$. In applying Gilbert's method we can see that the origin O' not only has no translatory motion in regard to fixed axes, but no motion whatever, relative to the latter. Therefore the case is of pure rotation about a fixed point O'; we have no function K to calculate. The potential function is, as usual, $U = -gz$. Now the kinetic energy consists of three terms:

$2\boldsymbol{T} = y'^2 + z'^2$ (this is the kinetic energy as if the axes x, y, z were absolute).

$2\boldsymbol{G} = (l^2 + y^2)\omega$ (this is the same as would be $T = \frac{1}{2}(mr^2)\omega$, or $I(\omega^2/2)$; which is the kinetic energy of rotation of a particle about the axis Z).

$\boldsymbol{V} = \omega ly'$ (this will be evident; ω is the instantaneous velocity, which, in this case happens to be permanent; ly' is the same as mvl, that is the moment of momentum of the particle (in case of a system we would have had the impulse axis or the resultant moment of momentum); the angle (ω, P) is clearly $= O$, since both axes coincide).

The total kinetic energy will therefore be

$$2T = y'^2 + z'^2 + (l^2 + y^2)\omega^2 + 2\omega ly'.$$

Finally the two Lagrange's equations, for z and y, will be

$$z'' = -g;$$

and

$$\frac{d}{dt}(y' + l\omega) - y\omega^2 = 0;$$

that is $y'' - \omega^2 y = 0$.

From the first equation we have

$$z = -\frac{gt^2}{2} + C_1 t + C_2;$$

from the second, $y = C_3 e^{\omega t} + C_4 e^{-\omega t}$; and these are the final equations of motion.

Example 2. *Foucault's gyroscope. A spinning gyroscope* (fig. 46) *is so arranged in its gimbals that its axis of spin can*

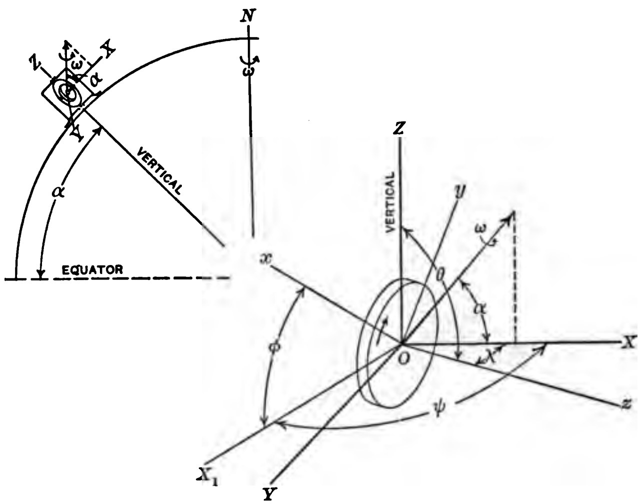


FIG. 46.

move only in a horizontal plane. The balance is such that the center of gravity of the wheel is exactly at the center of suspension, that is at the intersection of Z with the axis of spin; find the equations of motion under the action of the rotation of the earth, ω.

Let x, y, z, be the principal axes of the gyroscope; z is the axis of spin, and the moment of inertia about it is $= C$, as usual; $A = B$ are moments of inertia about x and y. Through the center, O, let us draw a set of fixed axes, X, Y, Z, so that X and Y are in the horizontal plane, in which the axis z is constrained to remain; furthermore, let the axis X be chosen to coincide with the projection, on that plane, of the axis ω of rotation of the earth; this is clearly indicated on the sketches. The angle α will evidently be the latitude. The Euler's angles will be:

ψ, giving the intersection $X_1 - O$ of the plane of the disk with the horizontal plane $X - Y$.

θ, the angle between ZO and zO; this is constant and $= 90°$ throughout the motion.

φ, the angle swept by any line xO, in the plane of the disk, in a given time; so that $d\varphi/dt$ is the velocity of spin.

Let us calculate the three parts of the kinetic energy, that is $\boldsymbol{T}$, $\boldsymbol{G}$ and $\boldsymbol{V}$:

$\boldsymbol{T}$ (that is considering the relative motion only). From the *motion of a body with one point fixed* we know that

$$2\boldsymbol{T} = A\omega_1^2 + B\omega_2^2 + C\omega_3^2;$$

or, in our case $(A = B)$,

$$2\boldsymbol{T} = A(\omega_1^2 + \omega_2^2) + C\omega_3^2,$$

but we have the following transformation formulae (see fig. 26)

$$\omega_1 = \frac{d\psi}{dt}\sin\theta\sin\varphi + \frac{d\theta}{dt}\cos\varphi,$$

$$\omega_2 = \frac{d\psi}{dt}\sin\theta\cos\varphi - \frac{d\theta}{dt}\sin\varphi,$$

$$\omega = \frac{d\psi}{dt}\cos\theta + \frac{d\varphi}{dt}.$$

Considering that $\theta = 90°$ we have

$$\omega_1 = \psi'\sin\varphi; \qquad \omega_2 = \psi'\cos\varphi; \qquad \omega_3 = \varphi';$$

whereby $2\boldsymbol{T} = A\psi'^2 + C\varphi'^2$.

$\boldsymbol{G}$ is the kinetic energy $I(\omega^2/2)$, of the rotation of the system x, y, z (including the gyroscope) about the instantaneous axis of rotation, which is parallel to the rotation of the earth (mention has been made before, why it is possible to transfer rotations from one point to another). But how can the moment of inertia I be calculated in reference to this axis ω? The ellipsoid of inertia with regard to principal axes is

$$Ax_1^2 + By_1^2 + Cz_1^2 = 1$$

(where x_1, y_1, z_1, are the current coordinates of the ellipsoid; see Bowser, Anal. Mech., pp. 443 and 444); or, in our case $A(x_1^2 + y_1^2) + Cz_1^2 = 1$. We also know that the moment of inertia about any line passing through the center of ellipsoid (for instance the line ω) will be

$$I = A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma;$$

where α, β, γ are the angles of the line with the principal axes. In our case $I = A(\cos^2 \alpha + \cos^2 \beta) + C \cos^2 \gamma$; now γ is the angle between ω and z, and can be immediately shown to be $\cos \gamma = \cos \alpha \sin \psi$ (by projecting ω upon z). Then, again, from the well-known formula of Anal. Geometry [see Bowser, Anal. Geom., p. 261, form (2)] we have

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1;$$

so that

$$\cos^2 \alpha + \cos^2 \beta = 1 - \cos^2 \gamma.$$

Substituting these expressions in our formula for I we have

$$I = A(1 - \cos^2 \alpha \sin^2 \psi) + C \cos^2 \alpha \sin^2 \psi;$$

hence

$$2\mathbf{G} = \omega^2[A + (C - A) \cos^2 \alpha \sin^2 \varphi].$$

V; this is the product of the instantaneous velocity of rotation (in our case, ω, the permanent rotation of the earth) by the impulse axis, and by the cosine of the angle between them:

$$\boldsymbol{V} = \omega \times P \times \cos (\omega, P).$$

The projections of the impulse axis on x, y, and z are (since $A = B$) $A\omega_1$, $A\omega_2$ and $C\omega_3$; the latter is (see transformation formulae) $= C\varphi'$; the other two components can be conveniently projected upon OX_1 (in the plane $x - y$), which gives $A(\omega_1 \cos \varphi - \omega_2 \sin \varphi)$, which, in view of transformation formulae, $= 0$; the same two components can also be projected upon O–Z, which gives $A(\omega_1 \sin \varphi + \omega_2 \cos \varphi)$, which in view of the same transformation formulae, $= A\psi'$. It

appears therefore that P is the resultant of $A\psi'$ (along Z) and of $C\varphi'$, (along z); hence its projection upon ω gives

$$V = \omega(A\psi' \sin\alpha + C\varphi' \cos\alpha \sin\psi);$$

therefore the total kinetic energy will be

$$2T = A\psi'^2 + C\varphi'^2 + \omega^2(A + (C - A)\cos^2\alpha \sin^2\psi) + 2\omega(A\psi' \sin\alpha + C\varphi' \cos\alpha \sin\psi).$$

There are no external forces, since U and K are both $= 0$, because of the suspension at the center of gravity.

Hence the two Lagrange's equations:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial\varphi'}\right) - \frac{\partial T}{\partial\varphi} &= 0, \\ \frac{d}{dt}\left(\frac{\partial T}{\partial\psi'}\right) - \frac{\partial T}{\partial\psi} &= 0. \end{aligned} \tag{1}$$

The first of these gives

$$\frac{d}{dt}(C\varphi' + \omega C \cos\alpha \sin\psi) = 0$$

or

$$\varphi' + \omega\cos\alpha \sin\psi = k. \tag{2}$$

Instead of expanding the second equation (1), the following artificial method will be used: The first equation (1) will be multiplied by φ', the second by ψ'; adding the products we have

$$\frac{d}{dt}\left(\varphi'\frac{\partial T}{\partial\varphi'} + \psi'\frac{\partial T}{\partial\psi'}\right) - \frac{d}{dt}\left(\varphi'\frac{\partial T}{\partial\varphi} + \psi'\frac{\partial T}{\partial\psi}\right) = 0.$$

The second term is simply $= dT/dt$; hence, integrating,

$$\varphi'\frac{\partial T}{\partial\varphi'} + \psi'\frac{\partial T}{\partial\psi'} = T + h.$$

Calculating separately $\partial T/\partial\varphi'$ and $\partial T/\partial\psi'$, multiplying them

by φ' and ψ' and adding, we have

$$A\psi'^2 + A\omega \sin \alpha \cdot \psi' + C\varphi'^2 + C\omega \sin \alpha \sin \psi \cdot \varphi' = T + h.$$

Multiplying by 2 and substituting $2T$ as originally calculated, we have after reduction

$$A\psi'^2 + C\varphi'^2 - \omega(A + (C - A) \cos^2 \alpha \sin^2 \psi) = h.$$

Now ω can be calculated from the fact that one revolution of the earth takes 24 hours or 86164 seconds; hence,

$$\omega = \frac{2\pi}{86164} = .000073$$

and the square, ω^2, of this will be insignificant; therefore, dropping the third term, we have

$$A\psi'^2 + C\varphi'^2 = h; \tag{3}$$

substituting φ' from (2) and again dropping the term involving ω^2, we have

$$A\psi'^2 - 2Ck\omega \cos \alpha \sin \psi = f$$

(f being a new constant, absorbing k^2). Here, assuming the angle $zOX = \lambda$, that is putting

$$\psi = \frac{\pi}{2} + \lambda$$

we have

$$\lambda'^2 - \frac{2Ck\omega}{A} \cos \alpha \cos \lambda = \frac{f}{A},$$

which means pendular motion with a period

$$T = 2\pi \sqrt{\frac{A}{Ck\omega \cos \alpha}}$$

(the reader, no doubt, knows that an equation

$$\left(\frac{d\varphi}{dt}\right)^2 = s(\cos \varphi - \alpha)$$

means pendular motion equivalent to a simple pendulum of length $l = 2g/s$).

Therefore the axis of the gyroscope will oscillate about the neutral position O–X; when the oscillations die out, due to frictional and other resistances, the axis of spin will place itself in the meridian plane, and *will indicate true north;* this is the principle of the gyro-compass.

On the other hand we were not limited in location of the plane XOY (of the motion of the axis of spin), which could have been directed in any manner. The calculations would have been the same, and the neutral position of the axis would always *tend* to coincide with the direction of the axis of the earth. Thus, if the axis of the gyroscope were free to move in a meridian plane, it would finally set itself parallel to the axis of the earth; its angle with the vertical would then indicate the latitude of the place.

CHAPTER V.

SMALL OSCILLATIONS.

I. General Remarks.—The first mention of vibratory motion was made in elementary kinematics (Bowser, Anal. Mech., p. 276) in connection with the problem of a particle, revolving in a circle, uniformly; it was shown that the linear acceleration of the particle, along a diameter, varies directly as the distance of the projection of the particle from the center. Indeed, if ω is the constant angular velocity and a the radius of the circle, we have $x = a \cos \theta$, where x is the distance of the projection of the particle from the center and θ is the angle of the diameter with the radius drawn from the center to the particle. Hence

$$\frac{dx}{dt} = - a \sin \theta \frac{d\theta}{dt} = - d\omega \sin \theta;$$

also

$$\frac{d^2x}{dt^2} = - a\omega \cos \theta \frac{d\theta}{dt} = - \omega^2 x,$$

that is the acceleration is proportional to the distance x, which is exactly what we tried to show. Note the negative sign, which shows that the acceleration is always directed *toward* the center. Later, at the very beginning of Dynamics (Bowser, Anal. Mech., p. 295) we had a somewhat similar problem: find the motion of a particle under action of an attractive force, varying as the distance of the particle from the center of force. Assuming the mass to be $= 1$, and that the coefficient of proportionality of the force to the distance is μ (that is, at the distance x the attraction is $= \mu x$), we have (force = mass times acceleration)

$$\frac{d^2x}{dt^2} = - \mu x, \qquad (1)$$

the negative sign meaning that the force opposes the motion. in other words showing the tendency of the force to *diminish* x;

Multiplying by $2dx$ and integrating we have

$$\left(\frac{dx}{dt}\right)^2 = \mu(a^2 - x^2). \tag{2}$$

If $x = a$ is the extreme travel of the particle, the distance from which the particle started when t was $= 0$, then $-(dx/\sqrt{a^2 - x^2}) = \sqrt{\mu}dt$, the negative sign being selected because x decreases as t increases. Integrating from $t = 0$ to $t = t$ we have $\cos^{-1}(x/a) = \sqrt{\mu}\cdot t$ or

$$t = \frac{1}{\sqrt{\mu}}\cos^{-1}\frac{x}{a}, \tag{3}$$

so that when $x = a$, $t = 0$; also when $x = 0$, $t = \pi/2\sqrt{\mu}$, which is the time of one travel from extreme position to the center. Having passed the center, where the acceleration is $= 0$ (from (1)) and where the velocity is maximum and $= a\sqrt{\mu}$ (from (2)), the particle will travel the same distance on the other side, until the velocity is again $= 0$ and the acceleration is again maximum and equal $-\mu a$. So that the particle will oscillate between the two positions $+a$ and $-a$; the time required for the particle to travel to the other extreme and back is evidently four times the value just found, $t = \pi/2\sqrt{\mu}$. It is called *complete* period of oscillation, T, so that

$$T = 2\pi\frac{1}{\sqrt{\mu}}, \tag{4}$$

or, since from (1) μ is equal to the acceleration divided by the extreme deflection, we have

$$T = 2\pi\sqrt{\frac{\text{extreme deflection}}{\text{acceleration}}}. \tag{5}$$

The form (5) should not mislead the reader: it is exactly equivalent to (4) and therefore shows that the period is

independent of the extreme deflection *taken by itself,* but depends only upon the value of the constant μ. A superficial inspection of (5) might suggest a certain doubt as to this being true, until the matter has been thought over. But the form (5) is useful in applications of the following nature:

1. A weight of 10 lbs. rests on a smooth horizontal plane, between two pins, to which it is connected by springs (Fig. 47). The springs are such that a displacement of 1 inch either

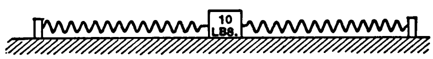


FIG. 47.

way calls out an elastic resistance of 2 lbs.; that of 2″ causes the resistance of 4 lbs., etc. Find the complete period of oscillation of the weight, if pulled to one side and let go. We know that the period will be independent of the displacement (alone), but to check this up we shall calculate the period for, say, a deflection of 1″ and that of 2″. If the displacement is 1″, the force called into play is 2 lbs. Now the mass of 10 lbs. is 10/32.2 and therefore the acceleration corresponding to the deflection of 1″ will be = force/mass = (2 × 32.2)/10; the deflection is = 1″ or 1/12 ft.; so that the complete period will be

$$T = 2\pi\sqrt{\frac{1 \times 10}{12 \times 2 \times 32.2}} \quad \text{or} \quad .73 \text{ sec.}$$

Now if the deflection is 2″, we have an identical result: here the deflection is 2/12 or 1/6 ft. The acceleration is = force/mass that is (4 × 32.2)/10; so that the period

$$T = 2\pi\sqrt{\frac{1 \times 10}{6 \times 4 \times 32.2}} = .73 \text{ sec.}$$

2. A 5-lb. weight hangs on a light spring; each additional pound stretches the spring one-half an inch. Find the com-

plete period of oscillation of the 5-lb. weight. Here we can use either the form (5) or the form (4). The latter can be applied as follows: to find μ we only have to remember that at the distance x the attraction is μx; for instance, if under 5-lb. load the stretch is x ft., then under 6 lbs. it will be $x + (1/24)$ ft.; or

$$5 = \mu x,$$

$$6 = \mu\left(x + \frac{1}{24}\right),$$

whence $\mu = 24$. Hence the period $T = 2\pi/\sqrt{24} = 1.28$ sec. In practical applications it is often required to find n, the number of complete oscillations *per second*, or N, the number of complete oscillations *per minute*. It will be clearly seen that

$$n = \frac{1}{T}, \quad \text{also that} \quad N = \frac{60}{T},$$

n is often called *frequency* of oscillation.

In the above problems and in the equation (1) we have shown that, if the force is exactly proportional to the displacement, the period is independent of the latter and the main problem of the theory of oscillations, which is *to find the period*, is then comparatively simple. All we have to remember (and this should be firmly committed to memory) is that

$$\frac{d^2x}{dt^2} = -\mu x$$

and

$$\left(\frac{dx}{dt}\right)^2 = \mu(a^2 - x^2)$$

mean exactly the same thing, namely simple harmonic motion $x = a \cos \sqrt{\mu}t$ (from (3)) with the period $T = 2\pi/\sqrt{\mu}$.

The reader must learn how instantly to write the period of any such oscillation as $(a^2 + b^2)\varphi'' + cg\varphi = 0$, which is

$T = 2\pi \sqrt{(a^2 + b^2)/cg}$; and in general it would be well for him to revise the usual methods of solution of the following important differential equation

$$a\frac{d^2x}{dt^2} + bx = 0 \quad \text{(same as (1))};$$

he would then see that the general solution of this equation is $x = A \sin \sqrt{(b/a)}t + B \cos \sqrt{(b/a)}t$, where A and B are constants. He would also see that this form can be easily transformed into $x = M \sin (\sqrt{(b/a)}t + N)$, where M is called *amplitude* of motion and N is a characteristic known as *phase;* so that the solution in either form means *pendular* or *sinusoidal* or *simple harmonic* motion.

How tremendously was increased the difficulty of the problem when the force is *not exactly* proportional to the displacement was shown later, in the problem of the simple pendulum (Bowser, Anal. Mech., p. 351). We found that the period of such a motion cannot be found at all by integration. The expression of dt can be reduced to Elliptic Integrals, which form the subject of a special branch of mathematics, highly interesting and most useful in applications.

However, dealing with the problem of the pendulum, we limited ourselves to the special case where the oscillations considered were very small (4° or 5° each way), which made it possible to introduce certain simplifying assumptions, to reject certain quantities, as insignificant, under our limitation, etc. In view of these assumptions we finally derived the well-known formula $T = 2\pi \sqrt{(l/g)}$, where T is the period of a complete (double) oscillation of the pendulum, and l is its length (in feet), g being the acceleration of gravity (= 32.2 ft. per sec.^{2}). For greater amplitudes of swinging this formula is, of course, absolutely worthless: the reader will readily imagine that the period can be anything up to (and including) infinity (for the case where the pendulum is allowed to swing 180° each way). The accompanying curve (Fig. 48) shows

the rapid increase of the coefficient before the square root, corresponding to increasing amplitudes. It should be remarked here that in problems on oscillatory motion it is often customary to express the solution not as period of oscillation,

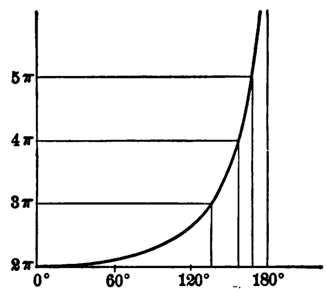


FIG. 48.

T, but as length of *equivalent simple pendulum*, l. The period always comes out to be some such expression as $T = 2\pi\sqrt{M}$; then all we have to do is to equate this to $T = 2\pi\sqrt{(l/g)}$, and to solve for l, which, then, is the required length of equivalent simple pendulum.

Now the problem of the simple pendulum was the first example of the exceedingly large and important class of problems known as *small oscillations*. Here we purposely limit ourselves to the small amplitudes, so that all arcs, velocities and displacements will be necessarily very small; their squares or products will be negligible; the sines of small angles will be replaced by the arcs proper, etc.

We shall illustrate the principle of small oscillations on the following examples:

Example 1.—A homogeneous hemisphere performs small oscillations on a rough (no sliding) horizontal plane; find the motion (Fig. 49). Let G be the center of gravity of the hemisphere, $CG = c$, and the radius $= r$; the angle NCA will be denoted θ. Since N is the instantaneous center of rotation we can take moments (due to rectifying action of

gravity) about it. Let k be the radius of gyration of the hemisphere about its center of gravity G. Then the radius of gyration about N will be calculated by taking into account

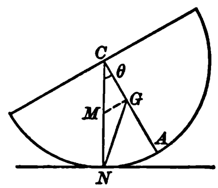


FIG. 49.

the square of the distance GN (see *Radius of Gyration*, Bowser, Anal. Mech., p. 435). Remembering now that *moment of inertia* $\times$ *angular acceleration* = *acting moment*, we can write

$$m(k^2 + GN^2)\theta'' = -mgc \sin \theta.$$

Now this equation is *absolutely exact;* but *since we limited ourselves to small oscillations we can simplify it by assuming that* GN *is practically* $= r - c$; *also that* $\sin \theta = \theta$, *the arc* θ *being very small.* Hence

$$[k^2 + (r - c)^2]\theta'' + gc\theta = 0,$$

so that the period is

$$T = 2\pi \sqrt{\frac{k^2 + (r - c)^2}{cg}}.$$

Inspection will show that $k^2 + c^2$ = square of the radius of gyration about C, which can be shown to be $(2/5)r^2$; also c can be found $= (3/8)r$, so that finally $T = 2\pi \sqrt{(26/15)(r/g)}$; this result will not hold good for any but small deflections from neutral position.

Example 2.—The theory of the Dynamic Balancing Machine, originated by the author, involves the following problem (Fig. 50): a heavy loaded beam, whose one end is hinged at A and the other is supported by the spring S, can perform small oscillations in a vertical plane. Find the period. The

mass of the beam is M, its moment of inertia about A is Mk^2, k being the radius of gyration, and the distance of its center of gravity from A is b. Let the initial deflection of the spring under its load be $= \delta$. The system is slightly disturbed

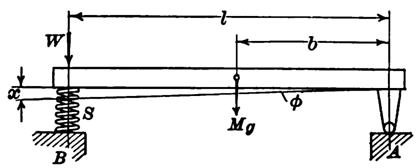


FIG. 50.

through a small amount x (and a small angular deflection φ) and let go. The force W acting on the spring when deflected through δ will result in a moment $Wl = Mgb$ about A; when deflected through another small amount x, this will increase by the amount

$$W\frac{\delta + x}{\delta}l - Wl = \frac{Wlx}{\delta},$$

which is then the acting moment, opposing the deflection, and can be written $= - mg(bx/\delta)$; now, since the acting moment $=$ moment of inertia $\times$ angular acceleration, we have

$$- Mg\frac{bx}{\delta} = Mk^2\varphi''.$$

Here the simplifying hypothesis is that, since the oscillations are small, we can assume $x = \varphi l$, whence $\varphi'' = x''/l$; substituting into the result just found we have

$$- Mg\frac{bx}{\delta} = Mk^2\frac{x''}{l},$$

so that $x'' + (gbl/k^2\delta)x = 0$, whence $T = 2\pi\sqrt{(k^2\delta/gbl)}$; writing this as $T = 2\pi\sqrt{(\delta/g)}\cdot\sqrt{(k^2/bl)}$ we can see that if we simply assumed the spring to vibrate under its own load, and due to its initial deflection δ, the result would be only,

$T = 2\pi\sqrt{\delta/g}$ (see similar problem above). But since the beam *oscillates* about A, this brings into play an important coefficient $\sqrt{k^2/bl}$, depending upon three other factors, characterizing the system. Since the beam in question is very heavy and the deflection δ is always very small, the oscillation cannot very well be any other but exceedingly small, so that, contrary to example 1, we do not have to emphasize the imposed limitation, the conditions of the problem take care of this automatically.

Later on these two easy problems will be solved by applying a different method.

In general, therefore, the problem of small oscillations involves a certain amount of experience in introducing simplifying assumptions.

We shall now show that by applying Lagrange's equations we can make this work much more uniform, almost mechanical.

Only oscillations *about the position of equilibrium* will be considered here; while in more advanced book is likewise included the problem of oscillation about steady motion.

II. Two important principles of equilibrium will now be established.

(a) When we have equilibrium, the potential function U is

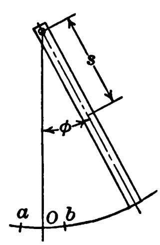


FIG. 51.

constant; or, in other words, partial derivatives of U with reference to any small displacement from the position of equilibrium are $= 0$. This will be readily understood: from the integral of kinetic energy we have $T = U + h$, where T

is the kinetic energy, U the potential function, and h is a constant (arbitrary). Therefore, in equilibrium, where the kinetic energy is $= 0$, the function U will naturally be constant. Thus for a pendulum (Fig. 51) in general $U = mgs \cos \varphi$, which for the position of equilibrium is simply $U = mgs$. A practical way of looking at it is that when the pendulum is at rest, then no infinitesimal lateral displacement will consume or produce any work at the expense of the force of gravity: all such infinitesimal displacements can be only horizontal, say Oa or Ob, and therefore the function U will remain constant.

(*b*) The second important principle, established by Lagrange is that for a position of *stable* equilibrium the potential function U must be not only constant, but at the same time *a maximum*. There exists a rigid proof of this proposition but we shall omit it here and the working of this principle will be explained by reference to the following typical problem:

A particle (Fig. 52) whose weight is M is attracted toward the centers A and B by means of springs AM and BM whose

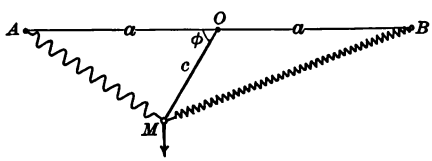


FIG. 52.

constants are λ and μ (that is the force of attraction is $\lambda \times AM$ and $\mu \times BM$ respectively); the particle is constrained to move about a fixed center O on radius c, O being midway between A and B. Find the position of equilibrium, also establish whether such equilibrium is stable or unstable?

Assuming φ to be the angle between AO and the present position OM, let us form the expression of virtual work of such a system for any small displacement of M in its rotation about O, say moving downward. We have already found

the expression of elementary work dW for this system, starting from the following (page 26)

$$dW = -\lambda AMdAM - \mu BMdBM + Md(c \sin \varphi);$$

here the last term of the second part is the virtual work due to gravity; the sign $-$ before the first two terms means, same as before, that the attraction opposes the increase of the distance from the attractive center; if there is any ambiguity as to whether this is *increase* or *decrease*, the geometric conditions will at once put this straight. Finally the following was found

$$dW = c[a(\mu - \lambda) \sin \varphi + M \cos \varphi]d\varphi = dU.$$

To find the maximum (or minimum) value of U we equate to O this expression of dU which gives

$$\tan \varphi = \frac{M}{a(\lambda - \mu)}.$$

To find whether this is maximum or minimum we differentiate once more

$$d^2W = c[a(\mu - \lambda) \cos \varphi - M \sin \varphi]d\varphi^2$$
$$= c[a(\mu - \lambda) \cot \varphi - M] \sin \varphi d\varphi^2;$$

considering the value of $\tan \varphi$ just found, the expression in square brackets [ ] will be always negative,

$$= -\frac{a^2}{M}(\mu - \lambda)^2 - M,$$

so that the value of $\sin \varphi$ will decide the condition of stability: if the value of φ, found from

$$\tan \varphi = \frac{M}{a(\lambda - \mu)},$$

is $< \pi$, the second derivative will be negative and there will be stable equilibrium. Otherwise, when φ is $> \pi$, the equilibrium is unstable.

We shall now somewhat enlarge our conception of potential function U. This has been defined as function of coordinates, such, that its derivatives with respect to coordinates give the value of corresponding forces, as for instance $\partial U/\partial x = X$ or $\partial U/\partial \varphi = \Phi$, etc. But the function U may have contained a certain constant, C, so that a more general way would have been to write $U + C$, where C may be *any constant,* and of course drops out in differentiation. For instance for a pendulum we simply had $U = mgs \cos \varphi$; this really should have been written $U = mgs \cos \varphi + C$ (where C is a perfectly arbitrary constant, positive, negative or zero); this was not done simply because we were always concerned only with derivatives of U, in which C does not enter; in other words we heretofore implied the assumption $C = 0$. But hereafter we shall assign a definite value to C: we shall select it equal to *minus the maximum value of* U, so that $U_{max} + C = U_{max} - U_{max} = 0$; in other words we shall select C so that the maximum value of U is $= 0$; and therefore all other values of U, viewed in this light, are negative. Consider for instance, an ordinary physical pendulum (Fig. 51). Here C is chosen $= - mgs$, so that the maximum value of U is zero; the increase of φ means decrease of the value of $(U + C)$, that is $mgs \cos \varphi - mgs$, until finally the value $\varphi = \pi$ is reached, in which case $U_{min} = - 2mgs$. This also means equilibrium, but it is *unstable.* For stable equilibrium the value of U must be maximum; and we have chosen this maximum value $= 0$; of course, so far as stability itself is concerned, any value of C would answer, but the object of selecting C so as to have U negative, or at the most $= 0$, will appear presently.

The physical pendulum is a system with one degree of freedom; as an example of a system with two degrees of freedom we shall consider a double pendulum (Fig. 53), for which the expression of U is $mg(l \cos \varphi + b \cos \varphi) + Mgs \times \cos \varphi + C$ (check this by considering the elevations of the center of gravity of each rod). Here, again, the constant C

will be chosen so that in the lowest position $U = 0$. This will be the stable equilibrium and it will be seen that any other position, corresponding to any change of φ or ψ will mean *negative* U, so that here again the condition of stable equilibrium means *maximum* U; when φ and ψ are both $= \pi$, we have unstable equilibrium, so that the slightest displacement

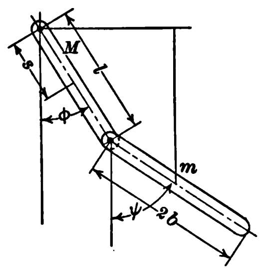


FIG. 53.

will upset such a system, but will not result in its coming back to the former position, as would be the case for stable equilibrium, where U is maximum (even if this maximum is but $= 0$). Limiting ourselves to the simplest problems, involving permanent constraints (so that the kinetic energy is merely a quadratic function of the *generalized velocities* q' etc.; see (4′) and (4″) Chapter II) we shall study the effect of *very small displacements*, such as q from the position of equilibrium. Only systems with *one* and with *two* degrees of freedom will be here considered.

III. Systems with one degree of freedom (for instance a pendulum). We shall consider the effect of giving such a system a slight displacement, q, from its position of equilibrium. Since, according to (4′), Chapter II, the expression of kinetic energy can be thrown into the following shape

$$2T = \Phi(q) \times q'^2, \qquad \text{(4′) Chapter II}$$

where $\Phi(q)$ is a function of q *only*, and constants (but does not involve q'), we can readily see that, (1), The kinetic energy

itself is very slight, since all velocities in $\Sigma(mv^2/2)$ are very small; also, (2), the displacement q is small, in accordance with the very nature of our problem; therefore, (3), the *generalized velocity* q' must likewise be small.

Now suppose that the function $\Phi(q)$ is developed by Maclaurin's formula (Bowser, Calculus, p. 83),

$$y = f(x) = f(0) + xf'(0) + \cdots,$$

so that

$$\Phi(q) = \Phi(0) + q\Phi'(0) + \cdots.$$

Now we shall neglect the term $q\Phi'(0)$ and will keep only the first term $\phi(0)$, so that instead of the above expression

$$2T = \Phi(q) \times q'^2$$

we shall have simply

$$2T = \Phi(0) \times q'^2$$

(dropping higher terms). Here $\Phi(0)$ is function of q no longer and this is precisely what we tried to accomplish. It is not always necessary to actually develop $\Phi(q)$ since the same result can be obtained by inspection; but the main object must be constantly kept in mind: the expression of T must be thrown into a form that does not involve q but only q'. The above expression $2T = \Phi(0)q'^2$ is generally written simply as

$$2T = aq'^2. \tag{1}$$

This is our first equation. The second can be obtained by developing the expression of U (which as we know is a function of q only) by Maclaurin's theorem

$$U = F(q) = F(0) + qF'(0) + \frac{q^2}{2}F''(0) + \cdots.$$

In this series $F(0) = 0$ because we agreed to adjust the constant C so that U is always $= 0$ in equilibrium (that is when q is $= 0$); likewise the term $qF'(0)$ is $= 0$, because in the posi-

tion of equilibrium (that is when $q = 0$) the value of U is maximum, so that the first derivative vanishes and the second derivative is *negative*. Finally, dropping higher terms, we have U in the following form

$$U = -\frac{bq^2}{2}, \tag{2}$$

where b is properly determined from $(q^2/2)F''(0) = -(bq^2/2)$. In other words we have outlined the possibility of expressing the potential function as *constant times the square of the independent coordinate* q. If this can be written down by inspection, the reader should by all means do so. He must satisfy himself that, in actual practice, if the deflections are small, and reckoned from the position of equilibrium as zero, then U will always come out to be a homogeneous quadratic function of q. Take for instance a weight on an elastic string: let the mass be m and the constant of the string $= \lambda$ (that is a stretch of x means resistance of λx lbs.). The potential function is found by computing the expression of elementary work

$$dW = -\lambda x dx + mgdx,$$

so that

$$U = -\frac{\lambda x^2}{2} + mgx + C,$$

which is *not* a quadratic function; but in position of equilibrium we have evidently $\lambda x = mg$. To avoid confusion let this value of x be designated δ, so that $\lambda\delta = mg$ or $\delta = mg/\lambda$. In other words what we had designated as x will now be made up of two parts: $x = \delta + z$, where z is the varying coordinate and δ simply characterizes the position of rest: $x = \delta$ when z is $= 0$.

Now substituting the value of x just found into the above expression of U (and remembering that $\lambda = mg/\delta$) we have

$$U = -\frac{mg}{2\delta}(\delta^2 + z^2) + C.$$

We have agreed once for all to select the constant in such a way that U is $= 0$ when $z = 0$ (position of rest). So that finally

$$U = -\frac{mg}{2\delta}z^2,$$

which is a quadratic function of the independent coordinate. This happens every time we limit ourselves to small displacements and reckon them from the position of rest. To return to the equations (1) and (2): applying Lagrange's method, instead of

$$\frac{d}{dt}\frac{\partial T}{\partial q'} - \frac{\partial T}{\partial q} = \frac{\partial U}{\partial q},$$

we have simply

$$aq'' = -bq, \tag{3}$$

which is the required differential equation of the small oscillation in question. The reader will see at once that it is the same as the well-known differential equation

$$\frac{d^2y}{dx^2} + r^2y^2 = 0,$$

of which the integral is $y = A \cos (rx + B)$; or, in our case,

$$q = \alpha \cos (rt + \beta), \tag{4}$$

where $r^2 = b/a$ and α and β are constants to be determined from initial conditions. As a rule we are not interested in them as much as we are in the constant r. The equation (4) means *periodic motion* with the period $T = 2\pi/r$. Indeed, to the value of t we can add any integer, n, multiplied by $2\pi/r$ and the result will be the same: thus

$$\begin{aligned} q = \alpha \cos (rt + \beta) &= \alpha \cos\left[r\left(\frac{2\pi}{r} + t\right) + \beta\right] \\ &= \alpha \cos\left[r\left(2\cdot\frac{2\pi}{r} + t\right) + \beta\right] \\ &= \alpha \cos\left[r\left(n\cdot\frac{2\pi}{r} + t\right) + \beta\right] = \text{etc.} \cdots; \end{aligned}$$

in other words q passes through the same value every $2\pi/r$ seconds. Now, *finding such periods of oscillation is precisely our problem.* We shall always mean *double* or *complete* periods.

So that, in order to apply Lagrange's method to the problem of small oscillations we have to (1) express T as constant times q'^2; and (2) express U as constant times q^2; after which the usual application of the method leads to an equation from which m can be seen at a glance, thus solving the problem.

By way of illustration we shall check the well-known formula of the simple pendulum, $T = 2\pi\sqrt{l/g}$ (Bowser, Anal. Mech., p. 353). We shall suppose that the length of the pendulum is l and its mass $= m$. Let the small deflection from the position of rest be φ, so that the current angular velocity will be φ'. This small deflection φ will be our generalized coordinate (same as q); and the kinetic energy will be

$$2T = mv^2 = m(l\varphi')^2 = ml^2\varphi'^2.$$

On the other hand the potential function U will be $U = mgz + C = -mgl(1 - \cos\varphi)$, or, developing $\cos\varphi$ by Maclaurin's formula (Bowser, Calculus, p. 86) and dropping higher terms we have $\cos\varphi = 1 - (\varphi^2/2)$; so that $U = -mgl(\varphi^2/2)$. Therefore our only Lagrange's equation will be

$$\frac{d}{dt}(ml^2\varphi'^2) = \frac{\partial}{\partial\varphi}\left(-mgl\frac{\varphi^2}{2}\right) \quad \text{or} \quad \varphi'' = -\frac{g}{l}\varphi,$$

whence $\varphi = \alpha\cos(\sqrt{(g/l)}t + \beta)$, — pendular motion of which m is $= \sqrt{g/l}$ so that the period will be $T = 2\pi/m = 2\pi\sqrt{l/g}$, which is the required formula.

For ready reference we shall give a few expansions, well known from calculus.

In general

$$y = f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + \cdots,$$

so that

$$\sin x = x - \frac{x^3}{6} + \cdots \text{ (for small values of } x \sin x = x),$$

$$\cos x = 1 - \frac{x^2}{2} + \cdots \text{ (often simply taken } = 1),$$

$$\tan x = x + \frac{x^3}{3} + \cdots \text{ (for small values of } x \tan x = x),$$

$$\sec x = 1 + \frac{x^2}{2} + \cdots \text{ (often simply taken } = 1).$$

Example 1.—Rocking hemisphere, Fig. 49. The height of the center of gravity G above the plane is $z = r - c \cos \theta$; so that the potential function will be $U = - mgz + C = mg \times (r - c \cos \theta) + C$; here C is selected so that the maximum value of U (when $\theta = 0$) is $= 0$; in other words the constant C is made $= mg(r - c)$ and finally $U = mgc(\cos \theta - 1)$ or, in view of the expanded formula of cos we have

$$U = - mgc \frac{\theta^2}{2}.$$

This corresponds to the equation (2) in the general theory: $U = - bq^2/2$. So far as the kinetic energy T is concerned we have $T = I(\omega^2/2)$ or $2T = I\theta'^2 = m(k^2 + (r - c)^2)\theta'^2$ (see how the moment of inertia was found before). This corresponds to the equation (1) of the general theory: $2T = aq'^2$. Hence, applying Lagrange's equation, or simply by (3) we have $aq'' = - bq$ or $m(k^2 + (r - c)^2)\theta'' = - mgc\theta$, the same equation as we had before.

In this and the following easy examples the answer could have been written almost without any calculations, which are given here only for the sake of making matters absolutely clear for the beginner.

Example 2.—A weight mg is suspended on a vertical coil spring of which the characteristic is λ (that is an elongation x calls into play a resistance of λx lbs.). Find the period of

the small oscillation of the weight when disturbed from its position of rest and let go.

Supposing that the small deflection from position of rest is z we have the following expression of kinetic energy

$$2T = mz'^2.$$

The potential function, as was shown above (see page 175), is

$$U = -\frac{mg}{2\delta}z^2$$

(where δ is the initial stretch due to load itself).

So that instead of

$$\frac{d}{dt}\frac{\partial T}{\partial z'} - \frac{\partial T}{\partial z} = \frac{\partial U}{\partial z}$$

we have $mz'' = -mgz/\delta$ which means that the period will be $T = 2\pi\sqrt{\delta/g}$, which we had before.

Example 3.—Oscillating beam, Fig. 50. Here we shall take as independent coordinate the angle φ, characterizing the deflection from horizontal position. Therefore the *generalized velocity* will be φ'; and if the moment of inertia is $= Mk^2$ then the expression of kinetic energy will be $2T = Mk^2\varphi'^2$. Now, the expression of elementary work will be

$$dW = -\lambda X dX + Mg\frac{b}{l}dX$$

or

$$U = -\frac{\lambda X^2}{2} + Mg\frac{b}{l}X + C,$$

where X is the total travel of the beam end from the no-load position of the spring.

When the beam is at rest the load on the spring is $= M(gb/l)$ and is resisted by $\lambda\delta$, where δ is the initial deflection of the spring; so that in the first place $\lambda = Mgb/l\delta$; and then $X = \delta + x$ where x is the characteristic small deflection of the end

of the beam from its neutral position, δ being merely a constant quantity. Substituting in the above expression of U we have

$$U = -\frac{\lambda}{2}(\delta + x)^2 + \frac{Mgb}{l}(\delta + x) + C$$

$$= -\frac{Mgb}{2l\delta}(\delta^2 + x^2) + \frac{Mgb\delta}{l} + C$$

or, with suitable adjustment of the constant C

$$U = -\frac{Mgb}{2l\delta}x^2,$$

now $x = l\varphi$, so that finally $U = -(Mgbl^2/2l\delta)\varphi^2$, whereby Lagrange's equation

$$\frac{d}{dt}\frac{\partial T}{\partial \varphi'} - \frac{\partial T}{\partial \varphi} = \frac{\partial U}{\partial \varphi},$$

or in our case

$$Mk^2\varphi'' = -M\frac{gbl^2}{l\delta}\varphi,$$

which means periodic motion with the period

$$T = 2\pi\sqrt{\frac{k^2\delta}{gbl}}$$

or $2\pi\sqrt{(\delta/g)}\ \overline{(k^2/bl)}$, which is the result we had before.

Example 4.—A particle of mass m is attached to two points A and B by two elastic strings whose constants are λ and μ and whose natural lengths are L and l. The distance between A and B is such that when the particle is in equilibrium the respective initial stretch of strings is, respectively, Δ and δ. Find the period of this small oscillation (Fig. 54).

The kinetic energy will be evidently given by

$$2T = mx'^2.$$

In order to find the potential function U let us find an expression of elementary work. If the distance x be taken as

independent coordinate, the pull exerted by the right string is $\mu(l + \delta - x)$; that exerted by the left string is $\lambda(l + \delta + \Delta$

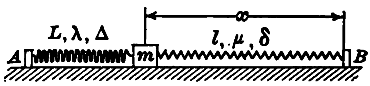


FIG. 54.

$- x)$. So that the elementary work will be

$$dW = \mu(l + \delta - x)dx - \lambda(l + \delta + \Delta - x)dx,$$

whence

$$U = -\frac{\mu}{2}(l + \delta - x)^2 + \frac{\lambda}{2}(l + \delta + \Delta - x)^2 + C$$

and the Lagrange's equation will be

$$mx'' = -x(\lambda + \mu) + A.$$

The solution of this differential equation will consist of a trigonometric term plus a certain constant. The reader is invited to investigate the nature of this constant. However this is quite immaterial for finding the period, which will be, according to rules given above,

$$T = 2\pi\sqrt{\frac{m}{\lambda + \mu}}.$$

The reader will check this result by forming new equations, where x has been measured from the position of equilibrium, and not from one of the fixed points.

IV. Systems with two degrees of freedom of which a typical illustration is the double pendulum, Fig. 53. Here each pendulum has its own period of oscillation, both periods being, in general, different from those corresponding to each pendulum swinging separately. The general treatment of the problem is similar to that of the previous case of one-degree freedom: all displacements are small and measured from the position of rest as zero. There will be two independent coordinates,

say q_1 and q_2, corresponding to the two-fold freedom of the system. The expression of kinetic energy for small displacements (and therefore small *generalized velocities* q_1' and q_2') will be given by a homogeneous quadratic function of generalized velocities, such as

$$2T = aq_1'^2 + 2bq_1q_2 + cq_2'^2. \tag{1}$$

Likewise the potential function U will be given by a homogeneous quadratic function of the generalized coordinates q_1 and q_2, such as

$$U = -\tfrac{1}{2}(\alpha q_1^2 + 2\beta q_1 q_2 + \gamma q_2^2). \tag{2}$$

This expression cannot very well contain any constants because $U = 0$ for $q_1 = 0$ and $q_2 = 0$; neither can it contain any terms of first degree in q_1 or q_2 because for $q_1 = 0$ and $q_2 = 0$ the potential function U is a maximum (compare with explanation for one-degree freedom).

Higher terms are neglected in expressions for both T and U.

Having T and U we can write down Lagrange's equations for both coördinates q_1 and q_2

$$\begin{aligned} \frac{d}{dt}\frac{\partial T}{\partial q_1'} - \frac{\partial T}{\partial q_1} &= \frac{\partial U}{\partial q_1}; \\ \frac{d}{dt}\frac{\partial T}{\partial q_2'} - \frac{\partial T}{\partial q_2} &= \frac{\partial U}{\partial q_2}; \end{aligned} \tag{3}$$

this will mean in view of (1) and (2)

$$\begin{aligned} aq_1'' + bq_2'' &= -(\alpha q_1 + \beta q_2), \\ bq_1'' + cq_2'' &= -(\beta q_1 + \gamma q_2). \end{aligned} \tag{4}$$

The usual method of solution of these linear differential equations of second order with constant coefficients consists of assuming

$$q_1 = p_1 \cos(rt + k); \qquad q_2 = p_2 \cos(rt + k), \tag{5}$$

where p_1, p_2, r and k are constants. What we are interested

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in are the values of r, of which as we shall presently see there will be two, r_1 and r_2, and from which we can determine the required periods $T_1 = 2\pi/r_1$ and $T = 2\pi/r_2$.

In order to find the constants r_1 and r_2 let us substitute the tentative solutions (5) into Lagrange's equations (4). We get

$$ap_1r^2 + bp_2r^2 = \alpha p_1 + \beta p_2, \tag{6}$$

$$bp_1r^2 + cp_2r^2 = \beta p_1 + \gamma p_2.$$

Eliminating the ratio p_1/p_2 we have an equation of second degree in r^2

$$(\alpha - ar^2)(\gamma - cr^2) - (\beta - br^2)^2 = 0.$$

Among the four roots of this equation, it can be shown that two values, r_1 and r_2 are positive. They are the required results.

Example 1.—Find periods of small oscillations of the double

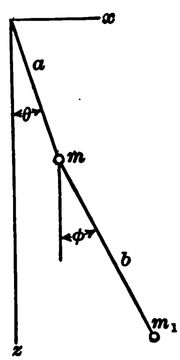

FIG. 55.

pendulum, Fig. 55, consisting of two masses, m and m_1, suspended by inextensible strings a and b.

Here the x, z coordinates of m and m_1 are

$$x = a \sin\theta; \qquad x_1 = a \sin\theta + b \sin\varphi;$$

$$z = a \cos\theta; \qquad z_1 = a \cos\theta + b \cos\varphi;$$

so that

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = v^2 = a^2\theta'^2;$$

also

$$\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dz_1}{dt}\right)^2 = a^2\theta'^2 + b^2\varphi'^2 + 2ab\theta'\varphi' \cos(\varphi - \theta)$$

and therefore the kinetic energy is given by

$$2T = ma^2\theta'^2 + m_1[a^2\theta'^2 + b^2\varphi'^2 + 2ab\theta'\varphi' \cos(\varphi - \theta)];$$

while the potential function will be found from the following expression of elementary work

$$dW = -\, mg \sin\theta d\theta - m_1 g \sin\theta d\theta - m_1 g \sin\varphi d\varphi.$$

(This can be written down at once by taking tangential components of the weight of each mass, m and m_1, and then by allowing the system a small displacement $d\theta$, maintaining φ constant; also by allowing a small displacement $d\varphi_1$ keeping θ constant; then adding the results.)

Hence the two Lagrange's equations,

$$\frac{d}{dt}\frac{\partial T}{\partial \theta'} - \frac{\partial T}{\partial \theta} = \frac{\partial U}{\partial \theta},$$

$$\frac{d}{dt}\frac{\partial T}{\partial \varphi'} - \frac{\partial T}{\partial \varphi} = \frac{\partial U}{\partial \varphi},$$

become

$$a(m + m_1)\theta'' + bm_1\varphi'' \cos(\varphi - \theta) - bm_1 \sin(\varphi - \theta)\varphi'^2 + (m + m_1)g \sin\theta = 0$$

and

$$a \cos(\varphi - \theta)\theta'' + b\varphi'' + a\theta'^2 \sin(\varphi - \theta) + g \sin\varphi = 0.$$

So far no simplifying assumption has been introduced and these equations are strictly correct, although much too difficult to integrate.

Limiting our problem to small oscillations we shall replace

the sines by corresponding arcs; also all cosines of small angles we shall simply put $= 1$; likewise all terms involving squares or products of velocities will be neglected. Hence

$$a(m + m_1)\theta'' + bm_1\varphi'' + (m + m_1)g\theta = 0,$$
$$a\theta'' + b\varphi'' + g\varphi = 0.$$

The first equation can be simplified by means of the second; so that finally

$$am\theta'' - m_1g\varphi + (m + m_1)g\theta = 0,$$
$$a\theta'' + b\varphi'' + g\theta = 0.$$

Assuming the usual tentative solution

$$\theta = p_1 \cos (rt + k) \quad \text{and} \quad \varphi = p_2 \cos (rt + k)$$

we finally have

$$amr^2 - (m + m_1)g + p_2m_1g = 0,$$
$$ar^2 + (br^2 - g)p_2 = 0.$$

From these equations we can find the two required *positive* values of $r : r_1$ and r_2. Considering the special case where $a = b$ and $m = m_1$ we have

$$ar^2 - 2g + p_2g = 0,$$
$$ar^2 + (ar^2 - g)p_2 = 0,$$

so that $r^2 = (g/a)(2 \pm \sqrt{2})$; in other words

$$r_1^2 = 3.414\frac{g}{a}, \qquad r_2^2 = .586\frac{g}{a},$$

whence the periods

$$T_1 = \frac{2\pi}{r_1} = .541 \times 2\pi\sqrt{\frac{a}{g}},$$
$$T_2 = \frac{2\pi}{r_2} = 1.307 \times 2\pi\sqrt{\frac{a}{g}}.$$

It will thus be seen that the upper pendulum will swing much

quicker and the lower much slower than what would be their natural rate $T_0 = 2\pi\sqrt{a/g}$.

Again, if we have a pendulum of the length $2a$, the rate will be still slower, $T_3 = 1.414 \times 2\pi\sqrt{a/g}$.

The investigation of the pendulum of Fig. 53 would lead to rather complicated results; the theory of such a pendulum is closely connected with the celebrated problem of *Bell and Clapper,* so ably given by Dr. Slocum in his book *The Theory and Practice of Mechanics.*

Example 2.—An investigation into the subject of *Dynamics of the Automobile* has brought the author to the following elementary problem: the Fig. 56 represents an imaginary

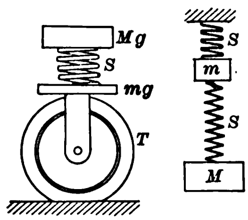

FIG. 56.

carriage, consisting of one tire T supporting a platform, representing the so-called "unsprung" weight mg. The "sprung" weight Mg rests on the spring S mounted on the platform. If the platform is steady, the natural period of oscillation will be

$$T = 2\pi\sqrt{\frac{\Delta}{g}},$$

where Δ is the initial deflection of the spring S under its load Mg; this we had before. As a matter of fact, however, the tire itself is elastic and the platform is susceptible to its own up-and down oscillations, although the elastic constant of the tire is much greater (the tire being stiffer) than that of the spring, say $\lambda_1 = 15\lambda$, where λ_1 is the constant of the tire and λ that of the spring. It will thus be seen that the problem

is reduced to a well-known problem of double spring, as shown on the right, Fig. 55.

We shall take as independent coordinates q_1 and q_2 deflections of each mass m and M from its neutral position. Then the kinetic energy is evidently given by $2T = mq_1'^2 + Mq_2'^2$.

In calculating the potential function U from the expression of elementary work we shall first consider the variation of q_1 alone, quite disregarding q_2; then the variation of q_2, although allowing for the fact that the variation of q_1 may have displaced the zero of q_2. For the upper spring s we have

$$U_1 = -\frac{\lambda_1 x_1^2}{2} + mgx_1 + C_1,$$

where, as we had before, x_1 is the total stretch, including the initial stretch δ and the deflection q_1 itself: $x_1 = \delta + q_1$. Simplifying and selecting proper constant we finally have

$$U_1 = -\frac{mg}{2\delta} q_1^2.$$

For the lower spring S we have

$$U_2 = -\frac{\lambda x_2^2}{2} + Mgx_2 + C_2.$$

It should be observed here that x_2 again consists of two terms: $x_2 = \Delta + q_2$, where Δ is the initial stretch of the spring S and q_2 is the small displacement. But the zero-position itself of M is dependent upon that of the support m of the spring S, which itself is characterized by its coordinate q_1. So that instead of x_2 we shall substitute $x_2 = \Delta + q_2 - q_1$, which after reductions and considering proper choice of C_2 gives

$$U_2 = -\frac{Mg}{2\Delta}(q_1^2 + q_2^2 - 2q_1q_2)$$

and the combined potential function is

$$U = -\frac{mgq_1^2}{2\delta} - \frac{Mg}{2\Delta}(q_1^2 + q_2^2 - 2q_1q_2),$$

so that the two Lagrange's equations in q_1 and q_2 will be as follows:

$$mq_1'' = -\lambda_1 q_1 + \lambda(q_2 - q_1),$$
$$Mq_2'' = -\lambda(q_2 - q_1).$$

Dividing the first equation by m and the second by M we can write them down as

$$\begin{aligned} q_1'' &= -aq_1 + b(q_2 - q_1), \\ q_2'' &= -c(q_2 - q_1). \end{aligned} \tag{α}$$

These are simultaneous differential equations of second order, linear and homogeneous. Their solutions will involve four constants.

It is not unreasonable to suppose that such a motion in general will be rather complicated. But let us see if we can find such a mode of combined oscillation in which each weight will perform simple harmonic vibrations (up and down) of the same period and phase, although without restrictions as to amplitudes. To test this out let us suppose that the solutions can be found in this form

$$\begin{aligned} q_1 &= p_1 \cos(rt + k), \\ q_2 &= p_2 \cos(rt + k), \end{aligned} \tag{β}$$

where q_1 and q_2 are the distances from the middle position of each weight; p_1 and p_2 are the amplitudes; the fact that r is the same meaning that we seek that mode of motion where both periods will be alike ($T = 2\pi/r$); and k being the same in both equations meaning that the phase is the same, in other words that both weights will be in their respective extreme positions, or, say, in their middle positions, at the same time. If we differentiate the equations (β) twice with respect to time and substitute in (α), the cosines cancel out and we have

$$\begin{aligned} -p_1r^2 &= -ap_1 + b(p_2 - p_1), \\ -p_2r^2 &= -c(p_2 - p_1). \end{aligned} \tag{γ}$$

Dividing by p_2 we can eliminate the ratio p_1/p_2, so that finally

$$r^4 - r^2(a + b + c) + ba = 0.$$

This is an equation of second degree in r^2, which can readily be shown to have two roots, $r_1{}^2$ and $r_2{}^2$, both real and positive. This means that we have obtained two distinct solutions of the form (β), each solution having its own constants p and k:

$$\left.\begin{aligned} q_{1a} &= p_1 \cos (r_1 t + k_1), \\ q_{1b} &= p_1' \cos (r_2 t + k_2), \\ q_{2a} &= p_2 \cos (r_1 t + k_1), \\ q_{2b} &= p_2' \cos (r_2 t + k_2), \end{aligned}\right\} \qquad (\delta)$$

these are the particular solutions of the differential equations (α). The general solution will consist of these particular integrals, superimposed:

$$\left.\begin{aligned} q_1 &= p_1 \cos (r_1 t + k_1) + p_1' \cos (r_2 t + k_2), \\ q_2 &= p_2 \cos (r_1 t + k_1) + p_2' \cos (r_2 t + k_2), \end{aligned}\right\} \qquad (\epsilon)$$

where the four constants p_1, p_1', k_1 and k_2 are arbitrary (determined from initial conditions); while the constants p_2 and p_2' are not independent but connected with p_1 and p_1', respectively, by either equation (γ), say the second, whereby

$$\frac{p_1}{p_2} = \frac{c - r_1{}^2}{c} \quad \text{and} \quad \frac{p_1'}{p_2'} = \frac{c - r_2{}^2}{c}. \qquad (\zeta)$$

It can be shown by theory of equations that this ratio will be positive for one and negative for the other of the roots r_1 and r_2. So that q_1 and q_2 may have the same sign or may be opposite in sign, while being of the same period. The initial conditions (that is the manner in which the system is started off) can be selected such that one of the constants, say p_1' is $= 0$; then the other corresponding constant p_2' is likewise $= 0$ (by ζ); and we have, instead of the general, more complicated

motion, simply

$$q_1 = p_1 \cos (r_1 t + k_1),$$
$$q_2 = p_2 \cos (r_1 t + k_1). \qquad (\vartheta)$$

It is easy to construct a model to demonstrate this sort of vibratory motion; both weights can be made to move in the same sense, or in opposite directions, although with the same period; only in the latter case the period will be much quicker than in the former. This difference in the periods can easily be seen from equations (ζ). Indeed, placing $p_1/p_2 = A$ we have (since this may be either $+$ or $-$),

$$r_1^2 = c + cA,$$
$$r_2^2 = c - cA,$$

so that the periods will be $T_1 = 2\pi/r_1$ and $T_2 = 2\pi/r_2$ respectively.

Example 3.—This was likewise suggested by the author's investigation into dynamics of the automobile. The car-body can have three main degrees of freedom and, accordingly, three modes of oscillation: vertical or plunging, angular about a longitudinal axis, or rolling; and angular about a transverse axis, or pitching. Assuming, however, that we have a two-dimensional car, possessing only length and height, let us find the periods of its two oscillations. The sketch relative to this much simplified problem is given in Fig. 57; a beam of

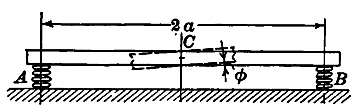

FIG. 57.

mass m is supported by two identical springs A and B, the constant of each spring being λ (in other words $\lambda\delta$ = half weight of beam). Such a beam possesses two degrees of freedom: it can vibrate vertically (independent coordinate z),

and can also oscillate about the transverse axis C (independent coordinate φ).

Therefore the expression of kinetic energy will be given by $2T = mz'^2 + mk^2\varphi'^2$, when k is the radius of gyration about C.

The potential function U will consist of two parts, one, U_1, depending upon the gravity; the other, U_2, upon the work of springs, due to a virtual angular displacement of the system.

Forming expressions of elementary work we have

$$dW_1 = -2\lambda(\delta + z)dz + mgdz,$$

hence

$$U_1 = -\lambda(\delta + z)^2 + mgz + C_1$$

or

$$U_1 = -\lambda z^2 = -\frac{mg}{\delta}z^2.$$

Also

$$dW_2 = -\lambda(\delta + x)dx + \lambda(\delta - x)dx;$$

hence

$$U_2 = -\frac{\lambda(\delta + x)^2}{2} - \frac{\lambda(\delta - x)^2}{2} + C_2$$

or

$$U_2 = -\lambda x^2 = -\frac{mg}{\delta}a^2\varphi^2;$$

so that

$$U = -\frac{mg}{\delta}(z^2 + a^2\varphi^2).$$

The Lagrange's equations will be

$$z'' = -\frac{2g}{\delta}z \qquad \text{and} \qquad \varphi'' = -\frac{2g}{\delta}\frac{a^2}{k^2}\varphi$$

and the periods will be $T_z = 2\pi\sqrt{\delta/2g}$ and $T_\phi = 2\pi\sqrt{(\delta/2g)}\cdot k/a$.

As a rule k is always less than a, therefore the free period of plunging is slower than that of pitching. But it is possible to artificially increase this value of k as well as to decrease a

to the limit which may be imposed by practical considerations, and thereby to make the period T_ϕ much slower than is the usual case. The object of this is to increase comfort in riding, which depends upon the period of pitching as well as upon the linear acceleration at the extreme points of the car, A and B, which may be even more remote than the distance of springs proper from the centerplane C.

If the oscillation is given by $\varphi = \alpha \cos (rt + \epsilon)$ where α is the amplitude and r can be calculated from $T = 2\pi/r$ and is

$$r = \sqrt{\frac{2g}{\delta}\,\frac{a}{k}}.$$

Now, the maximum angular acceleration can be found by deriving φ twice and by making $\cos (rt + \epsilon) = 1$; so that

$$\varphi_{max}'' = \alpha r^2 = \alpha \frac{2g}{\delta}\,\frac{a^2}{k^2}$$

and the maximum linear acceleration on the distance a from the center-plane will be proportional to a^3, hence the further importance of reducing a.

Example 4 (Loney).—A uniform rod (Fig. 58) of length $2a$,

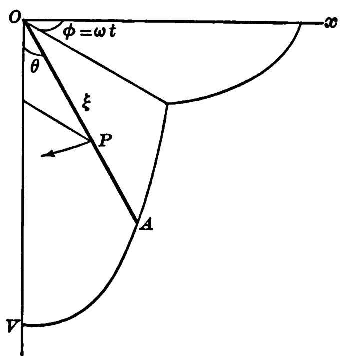

FIG. 58.

can turn freely about one end, which is fixed on a vertical axis. Initially the rod is inclined at an acute angle α to the

vertical and is set rotating about the vertical axis with angular velocity ω. Show that during the motion the rod is always inclined to the vertical at an angle which is $> \alpha$ or $< \alpha$ according as ω^2 is greater or less than $3g/4a \cos \alpha$, and that in each case its motion is included between the inclination α and

$$\cos^{-1} [- n + \sqrt{1 - 2n \cos \alpha + n^2}],$$

where $n = (a\omega^2 \sin^2 \alpha)/3g$. If the rod be disturbed, slightly, when revolving steadily at a constant inclination α, show that the period of the small oscillation will be

$$T = 2\pi\sqrt{\frac{4a \cos \alpha}{3g(1 + 3 \cos^2 \alpha)}}.$$

Let, at any time, t, the rod be inclined at the angle θ to the vertical; also let the plane through the rod and the vertical be characterized by the angle φ from the initial position. Consider an element of the rod, $d\xi$, at P, the distance from O being ξ. The mass of the whole rod being m, the unit-mass will be $m/2a$; and the mass of the element $d\xi$ will be $(d\xi/2a)m$. The circumferential velocity of P will be evidently $= \xi\omega \sin \theta$ (in its motion about the vertical). The velocity of P in its (possible) motion in the plane AOV will be $= \xi\theta'$, so that the kinetic energy of the element $d\xi$ will be given by

$$2T = \frac{d\xi}{2a} m(\xi^2\theta'^2 + \xi^2\omega^2 \sin^2 \theta)$$

and that of the whole rod will be given by

$$2T = \frac{m}{2a}(\theta'^2 + \omega^2 \sin^2 \theta) \int_0^{2a} \xi^2 d\xi = \frac{4ma^2}{3}(\theta'^2 + \omega^2 \sin^2 \theta).$$

Now the potential function U will be easily found from the elementary work of the center of gravity of the rod in displacing it up or down

$$U = mga \cos \theta + C.$$